Isidori I., Example 1.6.4

System of PDE (cannot be solved in a such way)

In[176]:=
```
Clear[gradu]
gradu = Grad[u[x, y, z, t], {x, y, z, t}]
DSolve[{gradu.{1, 0, 0, y} == 0, gradu.{0, 1, 0, x} == 0}, u[x, y, z, t], {x, y, z, t}]
```

Out[177]= $\{u^{(1,0,0,0)}[x, y, z, t], u^{(0,1,0,0)}[x, y, z, t], u^{(0,0,1,0)}[x, y, z, t], u^{(0,0,0,1)}[x, y, z, t]\}$

Out[178]= $\text{DSolve}[\{y\, u^{(0,0,0,1)}[x, y, z, t] + u^{(1,0,0,0)}[x, y, z, t] == 0,$
$x\, u^{(0,0,0,1)}[x, y, z, t] + u^{(0,1,0,0)}[x, y, z, t] == 0\}, u[x, y, z, t], \{x, y, z, t\}]$

Unfortunately, this cannot be solved

PDE of the system can be solved one by one

In[198]:=
```
ClearAll[x, u, gradu]
gradu = Grad[u[x, y, z, t], {x, y, z, t}];
```

In[200]:=
```
g1 = {1, 0, 0, y};
g2 = {0, 1, 0, x};
```

In[202]:=
```
DSolve[gradu.g1 == 0, u[x, y, z, t], {x, y, z, t}]
```

Out[202]= $\{\{u[x, y, z, t] \to C[1][y, z][t - x\, y]\}\}$

The solution ($C[1][y, z][t-\mathrm{xy}]$) of the first equation can be interpreted as a function of y, z and of $t-\mathrm{xy}$.
For example:

In[212]:=
```
u = (Sin[y z] + z) (t - x y);
Simplify[Grad[u, {x, y, z, t}].g1]
```

Out[213]= 0

In[218]:=
```
u = (Sin[y z] + z) Sin[(t - x y) z + y];
Simplify[Grad[u, {x, y, z, t}].g1]
```

Out[219]= 0

```
DSolve[gradu.g2 == 0, u[x, y, z, t], {x, y, z, t}]
```

$\{\{u[x, y, z, t] \to C[1][x, z][t - x\, y]\}\}$

From these solutions we can give a common solution.
Some examples: $u_1(x) = x_4 - x_1\, x_2$, $u_2(x) = x_3$

Isidori I., Example 4.1.4

In[224]:=
```
ClearAll[x, g, u]
x = {x1, x2, x3};
g = {Exp[x2], 1, 0};
gradu = Grad[u[x1, x2, x3], x];
DSolve[gradu.g == 0, u[x1, x2, x3], x]
```

Solve: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information.

Out[228]= $\{\{u[x1, x2, x3] \to C[1][x3][e^{x2} - x1]\}\}$

A solution for the PDE can be: $u(x) = e^{x_2} - x_1 + C(x_3)$

Example 1 (a)

```
In[229]:= x = {x1, x2, x3};
g = {0, 1, Sin[x1]};
gradu = Grad[u[x1, x2, x3], x];
DSolve[gradu.g == 0, u[x1, x2, x3], x]
Out[232]= {{u[x1, x2, x3] → C[1][x1][x3 - x2 Sin[x1]]}}
```

From this output you can give a function(s) $\phi_i(x)$ such that $L_g\, \phi_i(x) = 0$.

Example 1 (b)

```
x = {x1, x2, x3};
g = {0, 0, x2/(x1 + 1)};
gradu = Grad[u[x1, x2, x3], x];
DSolve[gradu.g == 0, u[x1, x2, x3], x]
{{u[x1, x2, x3] → C[1][x1, x2]}}
```

From this output you can give a function(s) $\phi_i(x)$ such that $L_g\, \phi_i(x) = 0$.

Example 1 (c)

```
x = {x1, x2, x3, x4};
g = {1/(x4 + 1), 0, x2/(x4 + 1), 0};
gradu = Grad[u[x1, x2, x3, x4], x];
DSolve[gradu.g == 0, u[x1, x2, x3, x4], x]
{{u[x1, x2, x3, x4] → C[1][x2, x4][-x1 x2 + x3]}}
```

From this output you can give a function(s) $\phi_i(x)$ such that $L_g\, \phi_i(x) = 0$.

Example 1 (d)

```
In[65]:= x = {x1, x2, x3, x4};
g = {1, 0, Exp[x4] - 1, 0};
gradu = Grad[u[x1, x2, x3, x4], x];
DSolve[gradu.g == 0, u[x1, x2, x3, x4], x]
Out[68]= {{u[x1, x2, x3, x4] → C[1][x2, x4][x1 - e^x4 x1 + x3]}}
```

From this output you can give a function(s) $\phi_i(x)$ such that $L_g\, \phi_i(x) = 0$.

Example 1 (e)

In[111]:=
```
x = {x1, x2, x3, x4};
g = {0, ArcTan[x3], 0, x1 ArcTan[x3]};
gradu = Grad[u[x1, x2, x3, x4], x];
DSolve[gradu.g == 0, u[x1, x2, x3, x4], x]
```

Out[114]=
```
{{u[x1, x2, x3, x4] → C[1][x1, x3][-x1 x2 + x4]}}
```

From this output you can give a function(s) $\phi_i(x)$ such that $L_g\,\phi_i(x) = 0$.

Example 1 (f)

```
x = {x1, x2, x3, x4};
g = {0, ArcTan[x3], 0, x1 ArcTan[x3]};
gradu = Grad[u[x1, x2, x3, x4], x];
DSolve[gradu.g == 0, u[x1, x2, x3, x4], x]
```

```
{{u[x1, x2, x3, x4] → C[1][x1, x3][-x1 x2 + x4]}}
```

From this output you can give a function(s) $\phi_i(x)$ such that $L_g\,\phi_i(x) = 0$.

Example 1 (g), continuous fermenter

In[317]:=

$$\mu = \frac{x2}{K2\,x2^2 + x2 + K1};$$

```
x = {x1, x2};
h = x2;
```

$$f = \left\{\mu\, x1,\ -\frac{\mu\, x1}{Y}\right\}$$

Out[320]=

$$\left\{\frac{x1\,x2}{K1 + x2 + K2\,x2^2},\ -\frac{x1\,x2}{\left(K1 + x2 + K2\,x2^2\right)\,Y}\right\}$$

In[321]:=

$$g = \left\{-\frac{x1}{V},\ \frac{Sf - x2}{V}\right\};$$

```
gradu = Grad[u[x1, x2], x];
DSolve[gradu.g == 0, u[x1, x2], x]
```

Out[323]=

$$\left\{\left\{u[x1, x2] \to C[1]\left[\frac{-Sf + x2}{x1}\right]\right\}\right\}$$

Example 1 (h)

```
x = {x1, x2, x3, x4};
```

```
f = {x1 x2 - x1^3, x1, -x3, x1^2 + x2};
g = {0, 2 + 2 x3, 1, 0};
h = x4;
gradu = Grad[u[x1, x2, x3, x4], x];
DSolve[gradu.g == 0, u[x1, x2, x3, x4], x]
```

$$\left\{\left\{u[x1, x2, x3, x4] \to C[1][x1, x4]\left[\frac{1}{2}\left(-x2 + 2\,x3 + x3^2\right)\right]\right\}\right\}$$

Example 1 (i)

```
In[390]:= x = {x1, x2, x3};
z = {z1, z2, z3};
f = {0, x1 + x2^2, x1 - x2};
g = {Exp[x2], Exp[x2], 0};
h = x3;
gradu = Grad[u[x1, x2, x3], x];
eqn = gradu.g == 0;
DSolve[eqn, u[x1, x2, x3], x]
```

Out[397]= $\{\{u[x1, x2, x3] \to C[1][x3][-x1 + x2]\}\}$

Example 1 (j)

```
In[374]:= x = {x1, x2, x3};
z = {z1, z2, z3};
f = {x3 (1 + x2), x1, x2 (1 + x1)};
g = {0, 1 + x2, -x3};
h = x1;
gradu = Grad[u[x1, x2, x3], x];
eqn = gradu.g == 0;
DSolve[eqn, u[x1, x2, x3], x]
```

Out[381]= $\{\{u[x1, x2, x3] \to C[1][x1][(1 + x2)\, x3]\}\}$

Example 2 (a), Controllability from LTI

In[424]:=
```
x = {x1, x2, x3};
z = {z1, z2, z3};
```
$$f = \begin{pmatrix} -x3\,\mathrm{Cos}[x1]^2 \\ -x2 + x3 \\ x2 - 2\,x3 + 2\,\mathrm{Tan}[x1] \end{pmatrix};$$
$$g = \begin{pmatrix} \mathrm{Cos}[x1]^2 \\ 1 \\ 2 \end{pmatrix};$$
```
adfg = Simplify[Grad[g, x].f - Grad[f, x].g];
adffg = Simplify[Grad[adfg, x].f - Grad[f, x].adfg];
```

In[434]:=
```
gradu = Grad[u[x1, x2, x3], x];
DSolve[(gradu.g)[[1]] == 0, u[x1, x2, x3], x]
DSolve[(gradu.adfg)[[1]] == 0, u[x1, x2, x3], x]
DSolve[(gradu.adffg)[[1]] == 0, u[x1, x2, x3], x]
```

Out[435]= {{u[x1, x2, x3] → C[1][x3 - 2 Tan[x1], x2 - Tan[x1]]}}

Out[436]= $\{\{u[x1, x2, x3] \to C[1][\frac{1}{2}(2\,x3 - \mathrm{Tan}[x1]), \frac{1}{2}(2\,x2 + \mathrm{Tan}[x1])]\}\}$

Out[437]= {{u[x1, x2, x3] → C[1][x3 + Tan[x1], x2 + 2 Tan[x1]]}}

Example 2 (b), Controllability from LTI

In[497]:=
```
x = {x1, x2, x3};
z = {z1, z2, z3};
```
$$f = \begin{pmatrix} \frac{-2\,x1^2 + 2\,x3 + x1\,(-2 + x2 + 2\,x3)}{2 + x2 + x3} \\ \frac{-2\,x2 + x1\,(2 + x3)}{2 + x1} \\ -x2 - x3 \end{pmatrix};$$
$$g = \begin{pmatrix} \frac{2\,(2 + x1)}{2 + x2 + x3} \\ 1 \\ 1 \end{pmatrix};$$
```
adfg = Simplify[Grad[g, x].f - Grad[f, x].g];
adffg = Simplify[Grad[adfg, x].f - Grad[f, x].adfg];
```

```
In[503]:= gradu = Grad[u[x1, x2, x3], x];
          DSolve[(gradu.g)[[1]] == 0, u[x1, x2, x3], x]
          DSolve[(gradu.adfg)[[1]] == 0, u[x1, x2, x3], x]
          DSolve[(gradu.adffg)[[1]] == 0, u[x1, x2, x3], x]
```

Out[504]= $\left\{\left\{u[x1, x2, x3] \to C[1]\left[-\frac{-2-x2-x3}{2\,(2+x1)}, -\frac{-4-4\,x2-x1\,x2+x1\,x3}{2\,(2+x1)}\right]\right\}\right\}$

Out[505]= $\left\{\left\{u[x1, x2, x3] \to C[1]\left[-\frac{2+x2+x3}{2+x1}, -\frac{4-2\,x2-2\,x1\,x2-x1\,x3}{2+x1}\right]\right\}\right\}$

Out[506]= $\left\{\left\{u[x1, x2, x3] \to C[1]\left[\frac{2\,(2+x2+x3)}{2+x1}, -\frac{-8-2\,x2+x1\,x2+2\,x1\,x3}{2+x1}\right]\right\}\right\}$

Example 3, Controllable manifold, fed-batch fermenter

$$\mu = \frac{x2}{K2\,x2^2 + x2 + K1};$$
```
x = {x1, x2, x3};
h = S;
```
$$f = \left\{\mu\,x1, -\frac{\mu\,x1}{Y}, 0\right\}$$
$$g = \left\{-\frac{x1}{x3}, \frac{Sf - x2}{x3}, 1\right\}$$
$$\left\{\frac{x1\,x2}{K1 + x2 + K2\,x2^2}, -\frac{x1\,x2}{(K1 + x2 + K2\,x2^2)\,Y}, 0\right\}$$
$$\left\{-\frac{x1}{x3}, \frac{Sf - x2}{x3}, 1\right\}$$

```
adfg = Simplify[Grad[g, x].f - Grad[f, x].g];
adfg // MatrixForm
```
$$\begin{pmatrix} \frac{x1\,(Sf-x2)\,(-K1+K2\,x2^2)}{(K1+x2+K2\,x2^2)^2\,x3} \\ -\frac{x1\,(Sf-x2)\,(-K1+K2\,x2^2)}{(K1+x2+K2\,x2^2)^2\,x3\,Y} \\ 0 \end{pmatrix}$$

```
adffg = Simplify[Grad[adfg, x].f - Grad[f, x].adfg];
adffg // MatrixForm
```
$$\begin{pmatrix} -\frac{x1^2\,(K1^2\,Sf-K2\,(1+K2\,Sf)\,x2^4-K1\,x2\,(-2\,Sf+x2-4\,K2\,Sf\,x2+4\,K2\,x2^2))}{(K1+x2+K2\,x2^2)^4\,x3\,Y} \\ \frac{x1^2\,(K1^2\,Sf-K2\,(1+K2\,Sf)\,x2^4-K1\,x2\,(-2\,Sf+x2-4\,K2\,Sf\,x2+4\,K2\,x2^2))}{(K1+x2+K2\,x2^2)^4\,x3\,Y^2} \\ 0 \end{pmatrix}$$

```
gradu = Grad[u[x1, x2, x3], x];
DSolve[gradu.g == 0, u[x1, x2, x3], x]
DSolve[gradu.adfg == 0, u[x1, x2, x3], x]
DSolve[gradu.adffg == 0, u[x1, x2, x3], x]
```

$$\left\{\left\{u[x1, x2, x3] \to C[1]\left[\frac{-Sf + x2}{x1}, x1\,x3\right]\right\}\right\}$$

$$\left\{\left\{u[x1, x2, x3] \to C[1][x3]\left[\frac{x1 + x2\,Y}{Y}\right]\right\}\right\}$$

$$\left\{\left\{u[x1, x2, x3] \to C[1][x3]\left[\frac{x1 + x2\,Y}{Y}\right]\right\}\right\}$$

We try to find a common solution for the three different PDE:

$$\text{Expand}\left[\left(\frac{-Sf + x2}{x1} + \frac{1}{Y}\right) * x1\,x3\right]$$

$$-Sf\,x3 + x2\,x3 + \frac{x1\,x3}{Y}$$

$$\text{Expand}\left[\left(\frac{x1 + x2\,Y}{Y} - Sf\right) * x3\right]$$

$$-Sf\,x3 + x2\,x3 + \frac{x1\,x3}{Y}$$

$$\text{Solve}\left[-Sf\,x3 + x2\,x3 + \frac{x1\,x3}{Y} == a, x3\right]$$

$$\left\{\left\{x3 \to \frac{a\,Y}{x1 - Sf\,Y + x2\,Y}\right\}\right\}$$

```
Plot3D[(a Y)/(x1 - Sf Y + x2 Y) /. {a → -4, Y → 0.5, Sf → 10},
 {x1, 0, 4}, {x2, 0, 4}, PlotRange → {{0, 4}, {0, 4}, {0, 35}}]
```