Advanced Nonlinear Control Methods: Theory and applications

Homework exercises (28th August, Tuesday)

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We consider input-affine nonlinear systems of the form

$$\begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases} \tag{1}$$

Example 1. Let

(a*) $f(x) = \begin{pmatrix} x_3 \sin(x_1) \\ x_3 - x_3 x_2 \cos(x_1) \\ x_1 + x_2 \end{pmatrix}$, $g(x) = \begin{pmatrix} 0 \\ 1 \\ \sin(x_1) \end{pmatrix}$, $h(x) = x_1 + x_3 - x_2 \sin(x_1)$.

Comment: very interesting zero dynamics.

(b) $f(x) = \begin{pmatrix} \frac{x_1^2}{x_1+1} \\ \frac{x_1+x_3}{x_1+1} - x_1 \\ \frac{x_2 x_3}{x_1+1} \end{pmatrix}$, $g(x) = \begin{pmatrix} 0 \\ 0 \\ \frac{x_2}{x_1+1} \end{pmatrix}$, $h(x) = x_1 + x_2$.

(c**) $f(x) = \begin{pmatrix} x_1 \sin(x_1) \\ -x_2 \sin(x_1) \\ x_1 + x_3 \\ \frac{x_1 x_2}{x_4+1} \end{pmatrix}$, $g(x) = \begin{pmatrix} \frac{1}{x_4+1} \\ 0 \\ \frac{x_2}{x_4+1} \\ 0 \end{pmatrix}$, $h(x) = x_3 - x_1 x_2$.

Let the nominal input be: $u^* = 1$, i.e. $u = u^* + \bar{u}$, then $\dot{x} = \overbrace{f(x) + g(x)u^*}^{\bar{f}(x)} + g(x)\bar{u}$.

(d*) $f(x) = \begin{pmatrix} x_1 \sin(x_1) \\ x_1 - x_4 \\ x_1 - x_3 \\ \frac{x_1 x_2}{x_4+1} \end{pmatrix}$, $g(x) = \begin{pmatrix} 1 \\ 0 \\ e^{x_4} - 1 \\ 0 \end{pmatrix}$, $h(x) = x_2$.

(e*) $f(x) = \begin{pmatrix} x_3 - \frac{x_1}{x_4+1} - x_1 e^{x_4} \\ x_2 e^{x_4} \\ -x_1 - x_3 \\ -x_4 - \frac{x_1 x_2}{x_4+1} \end{pmatrix}$, $g(x) = \begin{pmatrix} 0 \\ -\arctan(x_3) \\ 0 \\ -x_1 \arctan(x_3) \end{pmatrix}$, $h(x) = x_4 - x_1 x_2$.

(f*) $f(x) = \begin{pmatrix} x_1 x_3 \\ -x_3 \\ x_1 x_2 - x_3 \\ x_3(x_4 - x_1) \end{pmatrix}$, $g(x) = \begin{pmatrix} 0 \\ \arctan(x_3) \\ 0 \\ x_1 \arctan(x_3) \end{pmatrix}$, $h(x) = x_4 - x_1 x_2$,

$x^* = \begin{pmatrix} 2 & 0 & 0 & 1 \end{pmatrix}^T$.

(g) $f(x) = \begin{pmatrix} \mu(x_2)x_1 \\ \frac{-\mu(x_2)x_1}{Y} \end{pmatrix}, \quad g(x) = \begin{pmatrix} -\frac{x_1}{V} \\ \frac{S_F - x_2}{V} \end{pmatrix}, \quad h(x) = x_2, \quad \mu(x_2) = \dfrac{x_2}{K_2 x_2^2 + x_2 + K_1}.$

Continuous fermentation process $V = 4,\ S_F = 10,\ Y = 0.5,\ K_1 = 0.03,\ K_2 = 0.5$.

(h**) $f(x) = \begin{pmatrix} x_1 x_2 - x_1^3 \\ x_1 \\ -x_3 \\ x_1^2 + x_2 \end{pmatrix}, \quad g(x) = \begin{pmatrix} 0 \\ 2x_3 + 2 \\ 1 \\ 0 \end{pmatrix}, \quad h(x) = x_4,$

It is not obvious how to compute $x = \Phi^{-1}(z)$. Use `assumeAlso( z2+z3+1-z4^2 > 0 & x3 > -1 )`.

(i) $f(x) = \begin{pmatrix} 0 \\ x_1 + x_2^2 \\ x_1 - x_2 \end{pmatrix}, \quad g(x) = \begin{pmatrix} e^{x_2} \\ e^{x_2} \\ 0 \end{pmatrix}, \quad h(x) = x_3.$

(j) $f(x) = \begin{pmatrix} x_3(1 + x_2) \\ x_1 \\ x_2(1 + x_1) \end{pmatrix}, \quad g(x) = \begin{pmatrix} 0 \\ 1 + x_2 \\ -x_3 \end{pmatrix}, \quad h(x) = x_1.$

Tasks for each model (a-j):

1. Determine the relative degree (r) of the system.
2. Select an appropriate linearizing coordinate transformation and nonlinear feedback for this system. *Hint.*

$$\begin{array}{c|c} z_1 = h(x) & z_{r+1} = \phi_{r+1}(x) \\ \dots & \dots \\ z_r = L_f^{r-1} h(x) & z_n = \phi_n(x) \end{array} \qquad \text{new coordinates:} \quad z = \Phi(x) = \begin{pmatrix} h(x) \\ \dots \\ L_f^{r-1} h(x) \\ \phi_r(x) \\ \dots \\ \phi_n(x) \end{pmatrix}$$

   Try to choose functions $\phi_{r+1}(x)$, ..., $\phi_n(x)$ such that $L_g \phi_i(x) = 0$.
3. Determine the inverse transformation $x = \Phi^{-1}(z)$. If the system is exact linearizable (the relative degree r is equal to the order of the system n), you are not required to compute $\Phi^{-1}(z)$.
4. If the system is not exact linearizable, give the equation of the zero dynamics using the computed inverse mapping $\Phi^{-1}(z)$. Is the zero dynamics stable?
5. Stabilize the controllable (i.e. linear) part of the system.
6. Simulate the closed loop in Simulink.
   Comment: some models have an unstable zero-dynamics, nevertheless, you try to design a stabilizing controller, but do not be disappointed if you do not succeed.

Example 2. Compute the controllability manifold for the following models:

$$(\mathrm{a}^{**}) \quad f(x) = \begin{pmatrix} -x_3\cos^2(x_1) \\ x_3 - x_2 \\ 2\tan(x_1) + x_2 - 2x_3 \end{pmatrix}, \quad g(x) = \begin{pmatrix} \cos^2(x_1) \\ 1 \\ 2 \end{pmatrix}.$$

Comment: Requires some manual computations to find a common solution for

$$\langle \frac{\partial u}{\partial x}, g(x) \rangle = 0, \quad \langle \frac{\partial u}{\partial x}, \mathrm{ad}_f g(x) \rangle = 0, \quad \langle \frac{\partial u}{\partial x}, \mathrm{ad}_f^2 g(x) \rangle = 0,$$

$$(\mathrm{b}^{*}) \quad f(x) = \begin{pmatrix} \frac{-2x_1^2 + x_1(x_2+2x_3-2)+2x_3}{x_2+x_3+2} \\ \frac{x_1(x_3+2)-2x_2}{x_1+2} \\ -x_2 - x_3 \end{pmatrix}, \quad g(x) = \begin{pmatrix} \frac{2(x_1+2)}{x_2+x_3+2} \\ 1 \\ 1 \end{pmatrix}.$$

Comment: Relatively easy to find a common solution for

$$\langle \frac{\partial u}{\partial x}, g(x) \rangle = 0, \quad \langle \frac{\partial u}{\partial x}, \mathrm{ad}_f g(x) \rangle = 0, \quad \langle \frac{\partial u}{\partial x}, \mathrm{ad}_f^2 g(x) \rangle = 0,$$

Notations:
$\mathrm{ad}_f g(x) = [f, g]$ is the Lie bracket of $f(x)$ and $g(x)$.
$\mathrm{ad}_f^2 g(x) = [f, [f, g]]$ is the Lie bracket of $f(x)$ and $[f, g]$.

References

[Isidori, 1995] Isidori, A. (1995). *Nonlinear control systems.* Springer-Verlag London, 3 edition.

[Isidori, 1999] Isidori, A. (1999). *Nonlinear Control Systems II.* Springer-Verlag London, London, UK, UK.