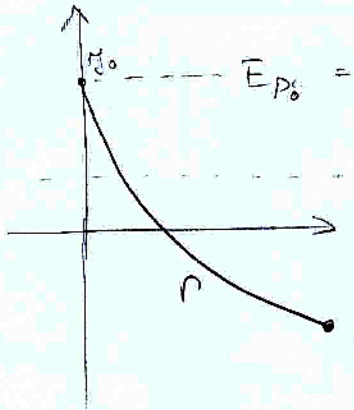
Brachistocrone   TISZTA

Adott egy anyagi pont $(0, y_0)$ pontban, adott egy görbe lejtő: vége (x_1, y_0).

Az anyagi pont $(0, y_0)$-ban $\frac{m v_0^2}{2}$ kinetikus energiával rendelkezik



$$E_{p_0} = mgy_0 \,;\quad E_{k_0} = \frac{m v_0^2}{2} \Rightarrow E_0 = mgy_0 + \frac{m v_0^2}{2}$$

$$\to y \;:\; E_k + E_p = E_0 \Rightarrow \frac{m v^2}{2} + mgy = E_0$$

$$E_p = mgy_1$$

$$\frac{m v^2}{2} = E_0 - mgy$$

$$v = \sqrt{\frac{2E_0}{m} - 2gy}$$

$$v = \sqrt{2g}\sqrt{\frac{E_0}{mg} - y}$$

$$T = \text{„}\frac{\text{dist}}{v}\text{"} = \int_\Gamma \frac{dl}{v} = \int_{x_0}^{x_1} \frac{\sqrt{1+y'^2}}{\sqrt{2g}\sqrt{\frac{E_0}{mg} - y}}\, dx$$

legyen $u = \frac{E_0}{mg} - y$

$u' = -y' \quad |^2$

$u'^2 = y'^2$

$$\boxed{T = \frac{1}{\sqrt{2g}} \int_{x_0}^{x_1} \frac{\sqrt{1+u'^2}}{\sqrt{u}}\, dx} \longrightarrow \min!$$

Jelentsünk egy általánosabb feladatot: $\int_{x_0}^{x_1} g(u)\sqrt{1+u'^2}\, dx \to \min$

$F(x, u, u') = F(u, u')$

↳ ez nincs!

$\Downarrow$

$$C = g(u)\sqrt{1+u'^2} - u' \underbrace{g(u)\frac{2u'}{2\sqrt{1+u'^2}}}_{\frac{\partial F}{\partial u'}} \Longleftarrow \boxed{F(u, u') - u' \frac{\partial F}{\partial u'}(u, u') = C}$$

$$C = \frac{g(u)}{\sqrt{1+u'^2}}\left(1 + u'^2 - u'^2\right)$$

Tehát:

$$\boxed{\frac{g(u)}{\sqrt{1+u'^2}} = C} \Longrightarrow 1 + u'^2 = \frac{g^2(u)}{C^2}$$

$$\Rightarrow u' = \pm\sqrt{\frac{g^2(u)}{C^2} - 1} \Longrightarrow \boxed{\frac{u'}{\pm\sqrt{\frac{g^2(u)}{C^2} - 1}} = 1}$$

Ennek jelentősége lesz.

Brachistocrone
Levezetés
Anal 3

$$\frac{u'}{\pm\sqrt{\frac{g^2(u)}{c^2}-1}} = 1 \quad \Big| \int dx \Longrightarrow \pm\int \frac{u'}{\sqrt{\frac{g^2(u)}{c^2}-1}}\,dx = x-b.$$

$du(x) = u'(x)\,dx$
röviden:
$du = u'\,dx$

$$\boxed{x-b = \pm\int \frac{du}{\sqrt{\frac{g^2(u)}{c^2}-1}}}$$

↳ Ezt kell majd megoldani

Brachistocrone: $g(u) = \frac{1}{\sqrt{u}} \overset{\text{ellenőr.}}{\Longrightarrow} F = g(u)\sqrt{1+u'^2} = \sqrt{\frac{1+u'^2}{u}}$ ✓

$$x-b = \pm\int\frac{du}{\sqrt{\frac{1}{c^2u}-1}} \overset{k:=\frac{1}{c^2}}{=} \pm\int \frac{du}{\sqrt{\frac{k-u}{u}}} = \pm\int\sqrt{\frac{u}{k-u}}\,du$$

Ha $\theta\in[0,\pi)$ ⇒ legyen +
Ha $\theta\in(\pi,2\pi]$ ⇒ legyen −

Legyen $u = k\sin^2\frac{\theta}{2}$ (*)
(változócsere)

láncszabály alapján

$$du = \frac{\partial u}{\partial\theta}\,d\theta = 2k\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cdot\frac{1}{2}\,d\theta$$

$$du = k\sin\frac{\theta}{2}\cos\frac{\theta}{2}\,d\theta$$

$\theta\in[0,2\pi] \Rightarrow \sin\frac{\theta}{2}\geq 0$

$$x-b = \pm\int\sqrt{\frac{k\sin^2\frac{\theta}{2}}{k-k\sin^2\frac{\theta}{2}}}\;k\sin\frac{\theta}{2}\cos\frac{\theta}{2}\,d\theta = \pm k\int\frac{\sin\frac{\theta}{2}}{\left|\cancel{\cos\frac{\theta}{2}}\right|}\sin\frac{\theta}{2}\cancel{\cos\frac{\theta}{2}}\,d\theta =$$

$$= k\int\sin^2\frac{\theta}{2}\,d\theta = k\int\frac{1-\cos\theta}{2}\,d\theta = \frac{k}{2}(\theta-\sin\theta)$$

Tehát:

$$x = b+\frac{k}{2}(\theta-\sin\theta)$$

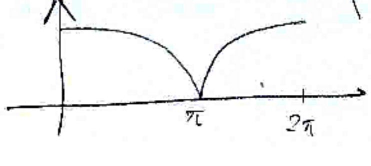
vált. csere

(*) $\Longrightarrow u = \frac{k}{2}(1-\cos\theta)$

$$u = \frac{E_0}{mg} - y$$

$$\Longrightarrow \boxed{\begin{aligned} x &= b+\frac{k}{2}(\theta-\sin\theta) \\ y &= \frac{E_0}{mg}-\frac{k}{2}(1-\cos\theta)\end{aligned}}$$

Megj: ha fordítva vesszük fel az előjeleket, akkor ez jött volna ki:
(valami hasonló)



$$\boxed{\begin{aligned} x &= b + \frac{k}{2}(\theta - \sin\theta) \\ y &= \frac{E_0}{mg} - \frac{k}{2}(1-\cos\theta) \end{aligned}}$$

$\theta \in [\theta_1, \theta_2]$

$$\left.\begin{aligned} x(\theta_1) &= x(t=0) = 0 \\ y(\theta_1) &= y(t=0) = y_0 \\ x(\theta_2) &= x(t=T) = x_1 \\ y(\theta_2) &= y(t=T) = y_1 \end{aligned}\right\} \text{Ezeket tudjuk}$$

Kérdés: $\theta_1, \theta_2, k, b = ?$

$$\boxed{\begin{aligned} y(\theta_1) &= \frac{E_0}{mg} - \frac{k}{2}(1-\cos\theta_1) = y_0 \\ x(\theta_1) &= b + \frac{k}{2}(\theta_1 - \sin\theta_1) = 0 \\ y(\theta_2) &= \frac{E_0}{mg} - \frac{k}{2}(1-\cos\theta_2) = y_1 \\ x(\theta_1) &= b + \frac{k}{2}(\theta_2 - \sin\theta_2) = x_1 \end{aligned}}$$

$$\frac{k}{2}(1-\cos\theta_1) = 0 \Rightarrow \underline{\theta_1 =}$$

$$\xrightarrow{\theta_1=0} \underline{b=0}$$

$$\left.\begin{aligned} -\frac{k}{2}(1-\cos\theta_2) &= y_1 - y_0 \\ \frac{k}{2}(\theta_2 - \sin\theta_2) &= x_1 \end{aligned}\right\} =$$

Tfh $v_0 = 0 \Rightarrow E_0 = mgy$

$$\underline{\frac{E_0}{mg} = y_0}$$

$$(*) \overset{\div}{\Rightarrow} \frac{1-\cos\theta_2}{\theta_2 - \sin\theta_2} = \frac{y_0 - y_1}{x_1}$$

$$\Downarrow$$

$$\text{fsolve}\Big(x_1(1-\cos\theta_2) - (y_0 - y_1)(\theta_2 - \sin\theta_2)\Big) \Rightarrow \underline{\theta_2}$$

$$\underline{k = \frac{2x_1}{\theta_2 - \sin\theta_2}}$$

Az fsolve ezt is meg tudja oldani!

Brachistochro
Levezetés
Anal 3