Isidori I., Example 1.6.4

System of PDE (cannot be solved in a such way)

```
In[176]:= Clear[gradu]
          gradu = Grad[u[x, y, z, t], {x, y, z, t}]
          DSolve[{gradu.{1, 0, 0, y} == 0, gradu.{0, 1, 0, x} == 0}, u[x, y, z, t], {x, y, z, t}]

Out[177]= {u^(1,0,0,0)[x, y, z, t], u^(0,1,0,0)[x, y, z, t], u^(0,0,1,0)[x, y, z, t], u^(0,0,0,1)[x, y, z, t]}

Out[178]= DSolve[{y u^(0,0,0,1)[x, y, z, t] + u^(1,0,0,0)[x, y, z, t] == 0,
            x u^(0,0,0,1)[x, y, z, t] + u^(0,1,0,0)[x, y, z, t] == 0}, u[x, y, z, t], {x, y, z, t}]
```

Unfortunately, this cannot be solved

PDE of the system can be solved one by one

```
In[198]:= ClearAll[x, u, gradu]
          gradu = Grad[u[x, y, z, t], {x, y, z, t}];

In[200]:= g1 = {1, 0, 0, y};
          g2 = {0, 1, 0, x};

In[202]:= DSolve[gradu.g1 == 0, u[x, y, z, t], {x, y, z, t}]

Out[202]= {{u[x, y, z, t] → C[1][y, z][t - x y]}}
```

The solution (C[1][y, z][t – xy]) of the first equation can be interpreted as a function of y, z and of t – xy.
For example:

```
In[212]:= u = (Sin[y z] + z) (t - x y);
          Simplify[Grad[u, {x, y, z, t}].g1]

Out[213]= 0

In[218]:= u = (Sin[y z] + z) Sin[(t - x y) z + y];
          Simplify[Grad[u, {x, y, z, t}].g1]

Out[219]= 0

          DSolve[gradu.g2 == 0, u[x, y, z, t], {x, y, z, t}]

          {{u[x, y, z, t] → C[1][x, z][t - x y]}}
```

From these solutions we can give a common solution.
Some examples: $u_1(x) = x_4 - x_1 x_2$, $u_2(x) = x_3$

Isidori I., Example 4.1.4

In[224]:=
```
ClearAll[x, g, u]
x = {x1, x2, x3};
g = {Exp[x2], 1, 0};
gradu = Grad[u[x1, x2, x3], x];
DSolve[gradu.g == 0, u[x1, x2, x3], x]
```

Solve: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information.

Out[228]= $\{\{u[x1, x2, x3] \to C[1][x3][e^{x2} - x1]\}\}$

A solution for the PDE can be: $u(x) = e^{x_2} - x_1 + C(x_3)$

Example 1 (a)

In[229]:=
```
x = {x1, x2, x3};
g = {0, 1, Sin[x1]};
gradu = Grad[u[x1, x2, x3], x];
DSolve[gradu.g == 0, u[x1, x2, x3], x]
```
Out[232]=
```
{{u[x1, x2, x3] → C[1][x1][x3 - x2 Sin[x1]]}}
```

From this output you can give a function(s) $\phi_i(x)$ such that $L_g\,\phi_i(x) = 0$.

Example 1 (b)

```
x = {x1, x2, x3};
g = {0, 0, x2/(x1 + 1)};
gradu = Grad[u[x1, x2, x3], x];
DSolve[gradu.g == 0, u[x1, x2, x3], x]
```
```
{{u[x1, x2, x3] → C[1][x1, x2]}}
```

From this output you can give a function(s) $\phi_i(x)$ such that $L_g\,\phi_i(x) = 0$.

Example 1 (c)

```
x = {x1, x2, x3, x4};
g = {1/(x4 + 1), 0, x2/(x4 + 1), 0};
gradu = Grad[u[x1, x2, x3, x4], x];
DSolve[gradu.g == 0, u[x1, x2, x3, x4], x]
```
```
{{u[x1, x2, x3, x4] → C[1][x2, x4][-x1 x2 + x3]}}
```

From this output you can give a function(s) $\phi_i(x)$ such that $L_g\,\phi_i(x) = 0$.

Example 1 (d)

In[65]:=
```
x = {x1, x2, x3, x4};
g = {1, 0, Exp[x4] - 1, 0};
gradu = Grad[u[x1, x2, x3, x4], x];
DSolve[gradu.g == 0, u[x1, x2, x3, x4], x]
```
Out[68]=
```
{{u[x1, x2, x3, x4] → C[1][x2, x4][x1 - e^x4 x1 + x3]}}
```

From this output you can give a function(s) $\phi_i(x)$ such that $L_g\,\phi_i(x) = 0$.

Example 1 (e)

In[111]:=
```
x = {x1, x2, x3, x4};
g = {0, ArcTan[x3], 0, x1 ArcTan[x3]};
gradu = Grad[u[x1, x2, x3, x4], x];
DSolve[gradu.g == 0, u[x1, x2, x3, x4], x]
```

Out[114]=
```
{{u[x1, x2, x3, x4] → C[1][x1, x3][-x1 x2 + x4]}}
```

From this output you can give a function(s) $\phi_i(x)$ such that $L_g\,\phi_i(x) = 0$.

Example 1 (f)

```
x = {x1, x2, x3, x4};
g = {0, ArcTan[x3], 0, x1 ArcTan[x3]};
gradu = Grad[u[x1, x2, x3, x4], x];
DSolve[gradu.g == 0, u[x1, x2, x3, x4], x]
```

```
{{u[x1, x2, x3, x4] → C[1][x1, x3][-x1 x2 + x4]}}
```

From this output you can give a function(s) $\phi_i(x)$ such that $L_g\,\phi_i(x) = 0$.

Example 2 (continuous fermentation process)

$$\mu = \frac{S}{K2\,S^2 + S + K1};$$

```
x = {X, S};
h = S;
```

$$f = \left\{\mu\,X,\ -\frac{\mu\,X}{Y}\right\}$$

$$\left\{\frac{S\,X}{K1 + S + K2\,S^2},\ -\frac{S\,X}{\left(K1 + S + K2\,S^2\right)\,Y}\right\}$$

$$g = \left\{-\frac{X}{V},\ \frac{Sf - S}{V}\right\}$$

```
gradu = Grad[u[X, S], x];
DSolve[gradu.g == 0, u[X, S], x]
```

$$\left\{-\frac{X}{V},\ \frac{-S + Sf}{V}\right\}$$

$$\left\{\left\{u[X, S] \to C[1]\left[\frac{S - Sf}{X}\right]\right\}\right\}$$

```
Phi2 = (S - Sf)/X;
eqn = {h == z1, Phi2 == z2}
sol = Solve[eqn, {X, S}]
```

$$\left\{S == z1, \frac{S - Sf}{X} == z2\right\}$$

$$\left\{\left\{X \to -\frac{Sf - z1}{z2}, S \to z1\right\}\right\}$$

```
f // MatrixForm // TeXForm
```

```
\left(
\begin{array}{c}
 \frac{S X}{\text{K1}+\text{K2} S^2+S} \\
 -\frac{S X}{Y \left(\text{K1}+\text{K2} S^2+S\right)} \\
\end{array}
\right)
```

```
g // MatrixForm // TeXForm
```

```
\left(
\begin{array}{c}
 -\frac{X}{V} \\
 \frac{\text{Sf}-S}{V} \\
\end{array}
\right)
```

```
h // TeXForm
```

S

Ex8, Isidori I Example 4.1.5

```
x = {x1, x2, x3, x4};
f = {x1 x2 - x1^3, x1, -x3, x1^2 + x2};
g = {0, 2 + 2 x3, 1, 0};
h = x4;
gradu = Grad[u[x1, x2, x3, x4], x];
DSolve[gradu.g == 0, u[x1, x2, x3, x4], x]
```

$$\left\{\left\{u[x1, x2, x3, x4] \to C[1][x1, x4]\left[\frac{1}{2}\left(-x2 + 2\,x3 + x3^2\right)\right]\right\}\right\}$$

```
Lfh = Grad[h, x].f
```

$x1^2 + x2$

```
Phi3 = 2 x3 + x3^2 - x2;
Phi4 = x1;
```

```
Grad[Phi3, x].g
```

0

```
eqn = {h == z1, Lfh == z2, Phi3 == z3, Phi4 == z4}
sol = Solve[eqn, {x1, x2, x3, x4}]
```

$$\{x4 == z1, x1^2 + x2 == z2, -x2 + 2\, x3 + x3^2 == z3, x1 == z4\}$$

$$\{\{x1 \to z4, x2 \to z2 - z4^2, x3 \to -1 - \sqrt{1 + z2 + z3 - z4^2}, x4 \to z1\}, \{x1 \to z4, x2 \to z2 - z4^2, x3 \to -1 + \sqrt{1 + z2 + z3 - z4^2}, x4 \to z1\}\}$$

```
dz = {z1, Grad[Lfh, x].f + Grad[Lfh, x].g u,
  Grad[Phi3, x].f + Grad[Phi3, x].g u, Grad[Phi4, x].f + Grad[Phi4, x].g u}
dz = dz /. sol[[2]]
```

$$\{z1, x1 + 2\, x1\, (-x1^3 + x1\, x2) + u\, (2 + 2\, x3), -x1 - x3\, (2 + 2\, x3), -x1^3 + x1\, x2\}$$

$$\{z1, z4 + 2\, z4\, (-z4^3 + z4\, (z2 - z4^2)) + u\, (2 + 2\, (-1 + \sqrt{1 + z2 + z3 - z4^2})), -z4 - (-1 + \sqrt{1 + z2 + z3 - z4^2})\, (2 + 2\, (-1 + \sqrt{1 + z2 + z3 - z4^2})), -z4^3 + z4\, (z2 - z4^2)\}$$

```
ZeroDynamics = dz[[3 ;; 4]] /. {z1 → 0, z2 → 0}
Eigenvalues[Grad[ZeroDynamics, {z3, z4}] /. {z3 → 0, z4 → 0}]
```

$$\{-z4 - (-1 + \sqrt{1 + z3 - z4^2})\, (2 + 2\, (-1 + \sqrt{1 + z3 - z4^2})), -2\, z4^3\}$$

$$\{-1, 0\}$$

```
f // MatrixForm // TeXForm
```

```
\left(
\begin{array}{c}
 \text{x1} \sin (\text{x1}) \\
 -\text{x2} \sin (\text{x1}) \\
 \text{x1}+\text{x3} \\
 \frac{\text{x1} \text{x2}}{\text{x4}+1} \\
\end{array}
\right)
```

```
g // MatrixForm // TeXForm
```

```
\left(
\begin{array}{c}
 \frac{1}{\text{x4}+1} \\
 0 \\
 \frac{\text{x2}}{\text{x4}+1} \\
 0 \\
\end{array}
\right)
```

```
h // TeXForm
```

\text{x3}-\text{x1} \text{x2}

Ex9, Isidori I Example 4.2.5

```
x = {x1, x2, x3};
f = {x3 (1 + x2), x1, x2 (1 + x1)};
g = {0, 1 + x2, -x3};
gradu = Grad[u[x1, x2, x3], x];
Lgu = gradu.g;
adfg = Grad[g, x].f - Grad[f, x].g;
Lgfu = gradu.adfg;
adffg = Grad[adfg, x].f - Grad[f, x].adfg;
Lgffu = gradu.adffg;
```

```
adfg // Simplify // MatrixForm
adffg // Expand // MatrixForm
IsidoriAdffg =
 {(1 + x2) (1 + 2 x2) (1 + x1) - x1 x3, x3 (1 + x2), -x3 (1 + x2) (1 + 2 x2) - 3 x1 (1 + x1)} //
   Expand // MatrixForm
```

$$\begin{pmatrix} 0 \\ x1 \\ -(1+x1)\,(1+2\,x2) \end{pmatrix}$$

$$\begin{pmatrix} 1 + x1 + 3\,x2 + 3\,x1\,x2 + 2\,x2^2 + 2\,x1\,x2^2 - x1\,x3 \\ x3 + x2\,x3 \\ -3\,x1 - 3\,x1^2 - x3 - 3\,x2\,x3 - 2\,x2^2\,x3 \end{pmatrix}$$

$$\begin{pmatrix} 1 + x1 + 3\,x2 + 3\,x1\,x2 + 2\,x2^2 + 2\,x1\,x2^2 - x1\,x3 \\ x3 + x2\,x3 \\ -3\,x1 - 3\,x1^2 - x3 - 3\,x2\,x3 - 2\,x2^2\,x3 \end{pmatrix}$$

```
DSolve[Lgu == 0, u[x1, x2, x3], x]
DSolve[Lgfu == 0, u[x1, x2, x3], x]
```

$$\{\{u[x1, x2, x3] \to C[1][x1][(1+x2)\,x3]\}\}$$

$$\left\{\left\{u[x1, x2, x3] \to C[1][x1]\left[\frac{x2 + x1\,x2 + x2^2 + x1\,x2^2 + x1\,x3}{x1}\right]\right\}\right\}$$

Common solution for $L_g\, h(x) = 0$ and $L_g\, L_f\, h(x) = 0$ is

```
h = x1
```

x1

```
f // MatrixForm // TeXForm
```

```
\left(
\begin{array}{c}
 \text{x1} \sin (\text{x1}) \\
 -\text{x2} \sin (\text{x1}) \\
 \text{x1}+\text{x3} \\
 \frac{\text{x1} \text{x2}}{\text{x4}+1} \\
\end{array}
\right)
```

```
g // MatrixForm // TeXForm
```

```
\left(
\begin{array}{c}
 \frac{1}{\text{x4}+1} \\
 0 \\
 \frac{\text{x2}}{\text{x4}+1} \\
 0 \\
\end{array}
\right)
```

```
h // TeXForm
```

```
\text{x3}-\text{x1} \text{x2}
```

Ex10.1, Controllability from LTI

$$A = \begin{pmatrix} 0 & 1 & 1 \\ -1 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix};$$

$$B = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix};$$

```
x = {x1, x2, x3};
z = {z1, z2, z3};
Phi = {ArcTan[x2], x1 + x3, x2 + x1}
```

```
{ArcTan[x2], x1 + x3, x1 + x2}
```

```
sol = Solve[Phi == z, x]
```

```
{{x1 → z3 - Tan[z1], x2 → Tan[z1], x3 → z2 - z3 + Tan[z1]}}
```

```
f = Simplify[Collect[Expand[Grad[Phi, x].A.x /. sol[[1]]], z]];
g = Simplify[Collect[Expand[Grad[Phi, x].B /. sol[[1]]], z]];
f // MatrixForm
g // MatrixForm
```

$$\begin{pmatrix} -z3\,\mathrm{Cos}[z1]^2 \\ -z2 + z3 \\ z2 - 2\,z3 + 2\,\mathrm{Tan}[z1] \end{pmatrix}$$

$$\begin{pmatrix} \mathrm{Cos}[z1]^2 \\ 1 \\ 2 \end{pmatrix}$$

```
adfg = Simplify[Grad[g, z].f - Grad[f, z].g];
adfg // MatrixForm
```

$$\begin{pmatrix} 2\,\mathrm{Cos}[z1]^2 \\ -1 \\ 1 \end{pmatrix}$$

```
adffg = Simplify[Grad[adfg, z].f - Grad[f, z].adfg];
adffg // MatrixForm
```

$$\begin{pmatrix} \mathrm{Cos}[z1]^2 \\ -2 \\ -1 \end{pmatrix}$$

```
gradu = Grad[u[z1, z2, z3], z];
DSolve[(gradu.g)[[1]] == 0, u[z1, z2, z3], z]
DSolve[(gradu.adfg)[[1]] == 0, u[z1, z2, z3], z]
DSolve[(gradu.adffg)[[1]] == 0, u[z1, z2, z3], z]
```

$\{\{u[z1, z2, z3] \to C[1][z3 - 2\,\mathrm{Tan}[z1], z2 - \mathrm{Tan}[z1]]\}\}$

$\{\{u[z1, z2, z3] \to C[1][\frac{1}{2}(2\,z3 - \mathrm{Tan}[z1]), \frac{1}{2}(2\,z2 + \mathrm{Tan}[z1])]\}\}$

$\{\{u[z1, z2, z3] \to C[1][z3 + \mathrm{Tan}[z1], z2 + 2\,\mathrm{Tan}[z1]]\}\}$

```
z3 - 2 Tan[z1] - (z2 - Tan[z1])
```

$-z2 + z3 - \mathrm{Tan}[z1]$

```
z3 - 1/2 Tan[z1] - (z2 + 1/2 Tan[z1])
```

$-z2 + z3 - \mathrm{Tan}[z1]$

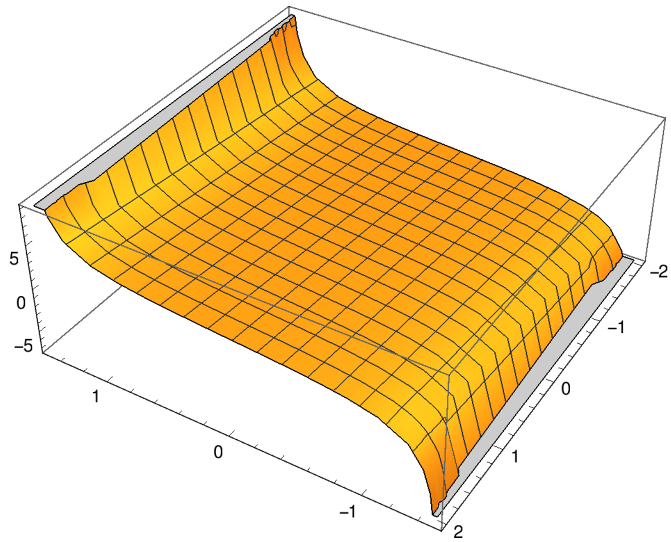
```
z3 + Tan[z1] - (z2 + 2 Tan[z1])
```

$-z2 + z3 - \mathrm{Tan}[z1]$

```
Solve[-z2 + z3 - Tan[z1] == a, z3]
```

$\{\{z3 \to a + z2 + \mathrm{Tan}[z1]\}\}$

```
Plot3D[a + z2 + Tan[z1] /. {a → 1}, {z1, -1.5, 1.5}, {z2, -2, 2}]
```




```
f // MatrixForm // TeXForm
```

```
\left(
\begin{array}{c}
 \text{x1} \sin (\text{x1}) \\
 -\text{x2} \sin (\text{x1}) \\
 \text{x1}+\text{x3} \\
 \frac{\text{x1} \text{x2}}{\text{x4}+1} \\
\end{array}
\right)
```

```
g // MatrixForm // TeXForm
```

```
\left(
\begin{array}{c}
 \frac{1}{\text{x4}+1} \\
 0 \\
 \frac{\text{x2}}{\text{x4}+1} \\
 0 \\
\end{array}
\right)
```

```
h // TeXForm
```

```
\text{x3}-\text{x1} \text{x2}
```

Ex10.2, Controllability from LTI

$$A = \begin{pmatrix} 0 & 1 & 1 \\ -1 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix};$$

$$B = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix};$$

```
x = {x1, x2, x3};
z = {z1, z2, z3};
```

$$\text{Phi} = \left\{\frac{x1 + x2}{x3 + 1}, x1 + x3, x2 + x3\right\};$$

```
sol = Solve[Phi == z, x]
```

$$\left\{\left\{x1 \to -\frac{-z1 - z2 - z1\,z2 + z3}{2 + z1},\ x2 \to -\frac{-z1 + z2 - z3 - z1\,z3}{2 + z1},\ x3 \to -\frac{z1 - z2 - z3}{2 + z1}\right\}\right\}$$

```
f = Simplify[Collect[Expand[Grad[Phi, x].A.x /. sol[[1]]], z]];
g = Simplify[Collect[Expand[Grad[Phi, x].B /. sol[[1]]], z]];
f // MatrixForm
g // MatrixForm
```

$$\begin{pmatrix} \frac{-2\,z1^2 + 2\,z3 + z1\,(-2 + z2 + 2\,z3)}{2 + z2 + z3} \\ \frac{-2\,z2 + z1\,(2 + z3)}{2 + z1} \\ -z2 - z3 \end{pmatrix}$$

$$\begin{pmatrix} \frac{2\,(2 + z1)}{2 + z2 + z3} \\ 1 \\ 1 \end{pmatrix}$$

```
adfg = Simplify[Grad[g, z].f - Grad[f, z].g];
adfg // MatrixForm
```

$$\begin{pmatrix} \frac{2 + z1}{2 + z2 + z3} \\ -1 \\ 2 \end{pmatrix}$$

```
adffg = Simplify[Grad[adfg, z].f - Grad[f, z].adfg];
adffg // MatrixForm
```

$$\begin{pmatrix} -\frac{2 + z1}{2 + z2 + z3} \\ -2 \\ 1 \end{pmatrix}$$

```
gradu = Grad[u[z1, z2, z3], z];
DSolve[(gradu.g)[[1]] == 0, u[z1, z2, z3], z]
DSolve[(gradu.adfg)[[1]] == 0, u[z1, z2, z3], z]
DSolve[(gradu.adffg)[[1]] == 0, u[z1, z2, z3], z]
```

$$\left\{\left\{u[z1, z2, z3] \to C[1]\left[-\frac{-2 - z2 - z3}{2\,(2 + z1)}, -\frac{-4 - 4\,z2 - z1\,z2 + z1\,z3}{2\,(2 + z1)}\right]\right\}\right\}$$

$$\left\{\left\{u[z1, z2, z3] \to C[1]\left[-\frac{2 + z2 + z3}{2 + z1}, -\frac{4 - 2\,z2 - 2\,z1\,z2 - z1\,z3}{2 + z1}\right]\right\}\right\}$$

$$\left\{\left\{u[z1, z2, z3] \to C[1]\left[\frac{2\,(2 + z2 + z3)}{2 + z1}, -\frac{-8 - 2\,z2 + z1\,z2 + 2\,z1\,z3}{2 + z1}\right]\right\}\right\}$$

$$\text{Simplify}\left[\frac{1}{5}\,(5\,z2 - \text{ArcSin}[z1]) + \frac{1}{5}\,(5\,z3 - \text{ArcSin}[z1])\right]$$

$$z2 + z3 - \frac{2\,\text{ArcSin}[z1]}{5}$$

$$\text{Simplify}\left[\frac{1}{7}\,(7\,z3 - 2\,\text{ArcSin}[z1]) + \frac{1}{7}\,(7\,z2 + \text{ArcSin}[z1])\right]$$

$$z2 + z3 - \frac{\text{ArcSin}[z1]}{7}$$

```
f // MatrixForm // TeXForm
```

```
\left(
\begin{array}{c}
 \text{x1} \sin (\text{x1}) \\
 -\text{x2} \sin (\text{x1}) \\
 \text{x1}+\text{x3} \\
 \frac{\text{x1} \text{x2}}{\text{x4}+1} \\
\end{array}
\right)
```

```
g // MatrixForm // TeXForm
```

```
\left(
\begin{array}{c}
 \frac{1}{\text{x4}+1} \\
 0 \\
 \frac{\text{x2}}{\text{x4}+1} \\
 0 \\
\end{array}
\right)
```

```
h // TeXForm
```

```
\text{x3}-\text{x1} \text{x2}
```

Ex11* Controllable manifold, fed-batch

fermenter

```
μ = x2/(K2 x2^2 + x2 + K1);
x = {x1, x2, x3};
h = S;
f = {μ x1, -μ x1/Y, 0}
g = {-x1/x3, (Sf - x2)/x3, 1}
```

$$\left\{\frac{x1\,x2}{K1+x2+K2\,x2^2},\ -\frac{x1\,x2}{\left(K1+x2+K2\,x2^2\right)Y},\ 0\right\}$$

$$\left\{-\frac{x1}{x3},\ \frac{Sf-x2}{x3},\ 1\right\}$$

```
adfg = Simplify[Grad[g, x].f - Grad[f, x].g];
adfg // MatrixForm
```

$$\begin{pmatrix} \frac{x1\,(Sf-x2)\,(-K1+K2\,x2^2)}{(K1+x2+K2\,x2^2)^2\,x3} \\ -\frac{x1\,(Sf-x2)\,(-K1+K2\,x2^2)}{(K1+x2+K2\,x2^2)^2\,x3\,Y} \\ 0 \end{pmatrix}$$

```
adffg = Simplify[Grad[adfg, x].f - Grad[f, x].adfg];
adffg // MatrixForm
```

$$\begin{pmatrix} -\frac{x1^2\,(K1^2\,Sf-K2\,(1+K2\,Sf)\,x2^4-K1\,x2\,(-2\,Sf+x2-4\,K2\,Sf\,x2+4\,K2\,x2^2))}{(K1+x2+K2\,x2^2)^4\,x3\,Y} \\ \frac{x1^2\,(K1^2\,Sf-K2\,(1+K2\,Sf)\,x2^4-K1\,x2\,(-2\,Sf+x2-4\,K2\,Sf\,x2+4\,K2\,x2^2))}{(K1+x2+K2\,x2^2)^4\,x3\,Y^2} \\ 0 \end{pmatrix}$$

```
gradu = Grad[u[x1, x2, x3], x];
DSolve[gradu.g == 0, u[x1, x2, x3], x]
DSolve[gradu.adfg == 0, u[x1, x2, x3], x]
DSolve[gradu.adffg == 0, u[x1, x2, x3], x]
```

$$\left\{\left\{u[x1, x2, x3] \to C[1]\left[\frac{-Sf+x2}{x1},\ x1\,x3\right]\right\}\right\}$$

$$\left\{\left\{u[x1, x2, x3] \to C[1][x3]\left[\frac{x1+x2\,Y}{Y}\right]\right\}\right\}$$

$$\left\{\left\{u[x1, x2, x3] \to C[1][x3]\left[\frac{x1+x2\,Y}{Y}\right]\right\}\right\}$$

Expand$\left[\left(\frac{-Sf + x2}{x1} + \frac{1}{Y}\right) * x1\ x3\right]$

$-Sf\ x3 + x2\ x3 + \frac{x1\ x3}{Y}$

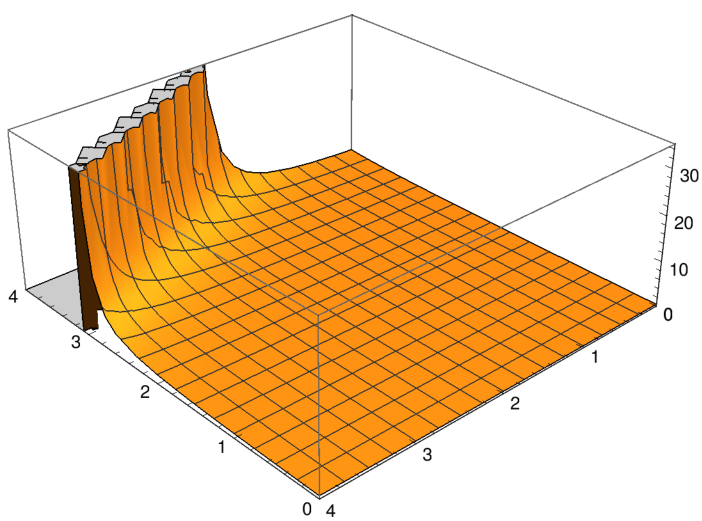
Expand$\left[\left(\frac{x1 + x2\ Y}{Y} - Sf\right) * x3\right]$

$-Sf\ x3 + x2\ x3 + \frac{x1\ x3}{Y}$

Solve$\left[-Sf\ x3 + x2\ x3 + \frac{x1\ x3}{Y} == a,\ x3\right]$

$\left\{\left\{x3 \to \frac{a\ Y}{x1 - Sf\ Y + x2\ Y}\right\}\right\}$

Plot3D$\left[\frac{a\ Y}{x1 - Sf\ Y + x2\ Y}\ /.\ \{a \to -4,\ Y \to 0.5,\ Sf \to 10\},\right.$
$\left.\{x1, 0, 4\}, \{x2, 0, 4\}, PlotRange \to \{\{0, 4\}, \{0, 4\}, \{0, 35\}\}\right]$



f // MatrixForm // TeXForm

```
\left(
\begin{array}{c}
 \text{x1} \sin (\text{x1}) \\
 -\text{x2} \sin (\text{x1}) \\
 \text{x1}+\text{x3} \\
 \frac{\text{x1} \text{x2}}{\text{x4}+1} \\
\end{array}
\right)
```

g // MatrixForm // TeXForm

```
\left(
\begin{array}{c}
 \frac{1}{\text{x4}+1} \\
 0 \\
 \frac{\text{x2}}{\text{x4}+1} \\
 0 \\
\end{array}
\right)
```

h // TeXForm

```
\text{x3}-\text{x1} \text{x2}
```