Nonlinear feedback design providing passivity

March 18, 2017

1 Theoretical background

Kérdés:

> 1. verzio
> (Our aim)/(The task) is to find an appropriate output function $h(x)$ for a nonlinear system $\dot{x} = f(x, u)$, such that, the zero dynamics of the feedback linearized system is stable using the designed output function.
> 2. verzio
> It is given an nonlinear system $\dot{x} = f(x, u)$, and we are looking for an output function $h(x)$, for which the exact feedback linearization will give/produce/entail a locally stable zero dynamics.

The problem is feasible (there exists a suitable $h(x)$) if the system accompanied/supplemented with this output law $h(x)$ is feedback equivalent with a passive system, in other word, if there exist $F(x), G(x)$ functions, s.t. the system

$$\begin{aligned} \dot{x} &= f(x, u) \qquad u = F(x)x + G(x)r \\ y &= h(x) \end{aligned} \tag{1}$$

is passive, more exactly, there exists a positive definite storage function $V(x)$ satisfying the inequality:

$$\dot{V}(x) \leq y^T r + r^T y \tag{2}$$

2 Output selection using LMI conditions

The nonlinear model representation is given in the following form:

$$\dot{x} = A(x)x + B(x)u \tag{3}$$

We are looking for some matrix functions $C(x)$, $F(x)$, $G(x)$:

$$u = F(x)x + G(x)r \tag{4}$$
$$y = C(x)x \tag{5}$$

such that the closed loop system

$$\begin{aligned} \dot{x} &= \widehat{A}(x)x + \widehat{B}(x)r \qquad & \widehat{A}(x) &= A(x) + B(x)F(x) \\ y &= C(x)x & \widehat{B}(x) &= B(x)G(x) \end{aligned} \tag{6}$$

will be passive, i.e there exists a storage function $V(x) = x^T P(x)x$ with a symmetric $P(x)$ s.t.

$$\dot{V}(x) \leq y^T r + r^T y \tag{7}$$

2.1 Passivity as an LMI condition

For the sake of simplicity we omit the arguments (x). The derivative of the storage function ca be derived as follows:

$$\begin{aligned} \dot{V} &= \dot{x}^T Px + x^T P\dot{x} + x^T \dot{P}x \\ &= (\widehat{A}x + \widehat{B}r)^T Px + x^T P(\widehat{A}x + \widehat{B}r) + x^T \dot{P}x \\ &= x^T(\dot{P} + \widehat{A}^T P + P\widehat{A})x + r^T \widehat{B}^T Px + x^T P\widehat{B}r \\ &= x^T(\dot{P} + A^T P + F^T B^T P + PA + PBF)x \\ &\quad + r^T G^T B^T Px + x^T PBGr \\ &= \begin{bmatrix} x \\ r \end{bmatrix}^T \begin{bmatrix} \dot{P} + A^T P + F^T B^T P + PA + PBF & PBG \\ G^T B^T P & 0 \end{bmatrix} \begin{bmatrix} x \\ r \end{bmatrix} \end{aligned} \tag{8}$$

The right-hand side of equation (7) can be rewritten as a quadratic form:

$$y^T r + r^T y = x^T C^T r + r^T Cx = \begin{bmatrix} x \\ r \end{bmatrix}^T \begin{bmatrix} 0 & C^T \\ C & 0 \end{bmatrix} \begin{bmatrix} x \\ r \end{bmatrix} \tag{9}$$

Using the above formulas the inequality of equation (7) can be expressed as:

$$\begin{bmatrix} x \\ r \end{bmatrix}^T \begin{bmatrix} \dot{P} + A^T P + F^T B^T P + PA + PBF & PBG - C^T \\ G^T B^T P - C & 0 \end{bmatrix} \begin{bmatrix} x \\ r \end{bmatrix} \leq 0 \tag{10}$$

This can be converted into a *sufficient* matrix inequality:

$$\begin{bmatrix} \dot{P} + A^T P + F^T B^T P + PA + PBF & PBG - C^T \\ G^T B^T P - C & 0 \end{bmatrix} \leq 0, \ \forall x \in \mathcal{X}$$

We assume, that $P(x)$ is invertible then we denote $Q(x) = P(x)^{-1}$. If we multiply both sides of the inequality by $[Q, I]_{\text{diag}}$, we obtain the following LMI:

$$\begin{bmatrix} Q\dot{P}Q + QA^T + QF^T B^T + AQ + BFQ & BG - QC^T \\ G^T B^T - CQ & 0 \end{bmatrix} \leq 0 \tag{11}$$

By differentiating the identity $PQ = I$, we get that $Q\dot{P}Q = -\dot{Q}$. Furthermore, we introduce the following notations: $M = FQ$, $N = CQ$, both of them are matrix function with respect to the variables (x). Using these notation, we can reach to a parameter dependent LMI:

$$\begin{bmatrix} -\dot{Q} + QA^T + M^T B^T + AQ + BM & BG - N^T \\ G^T B^T - N & 0 \end{bmatrix} \leq 0 \tag{12}$$

$$\begin{bmatrix} -\dot{Q} + QA^T + M^T B^T + AQ + BM & BG - N^T \\ G^T B^T - N & 0 \end{bmatrix} \leq 0 \tag{13}$$

where A, B are known variables, Q, M, N, G are unknown matrix functions of x. We cannot state that the LMI is linear in x, accordingly, the feasibility should be checked on an appropriately dense grid in $\mathcal{X}$: $\mathcal{X}_{grid} = \left\{ x_i \in \mathcal{X}, i = \overline{1, N} \right\}$, where $\mathcal{X}$ is a subset of the state space containing the possible values of the state vector.

Let us denote the left-hand side of equation (12) by $\Lambda(x)$. The SDP problem is reduced to:

$$\Lambda(x) \leq 0 \ , \quad \forall x \in \mathcal{X}_{grid} \tag{14}$$

As a result, we obtain the matrices Q, M, N, G from which C, F and P can be calculated.

2.2 Nonlinear feedback design

The feedback linearization will be processed on the original system with the previously generated output function, which will ensure that the zero dynamics of the linearized system will be stable on that region, on which the feasibility of the LMI (ref) was checked. Let us introduce the following notations on the original system:

$$\begin{aligned} \dot{x} &= A(x)x + B(x)r &&= f(x) + g(x)r \\ y &= C(x)x &&= h(x) \end{aligned} \tag{15}$$

Using the obtained output function, we are going to design a nonlinear feedback law as presented in [Isidori]. First of all, a coordinates transformation is defined:

$$\begin{aligned} z_1 &= h(x) \\ z_2 &= L_f h(x) \\ &\dots \\ z_r &= L_f^{r-1} h(x) \\ z_{r+1} &= \Phi_{r+1}(x) \\ &\dots \\ z_n &= \Phi_n(x) \end{aligned}$$

where r is the relative degree of the system. In the new coordinates the differential equation will look like the following:

$$\begin{aligned} \dot{z}_1 &= z_2 \\ &\dots \\ \dot{z}_{r-1} &= z_r \\ \dot{z}_r &= b(z) + a(z)v \\ \left.\begin{aligned} \dot{z}_{r+1} &= q_{r+1}(z) + p_{r+1}(z)v \\ &\dots \\ \dot{z}_n &= q_n(z) + p_n(z)v \end{aligned}\right\} &\text{ zero dynamics} \end{aligned}$$

if $v = \frac{1}{a(z)}\Big(-b(z) + v\Big)$, then

$$
\left.
\begin{aligned}
\dot{z}_1 &= z_2 \\
&\cdots \\
\dot{z}_{r-1} &= z_r \\
\dot{z}_r &= v
\end{aligned}
\right.
$$

$$
\left.
\begin{aligned}
\dot{z}_{r+1} &= q_{r+1}(z) + \frac{p_{r+1}(z)}{a(z)}\Big(-b(z)+v\Big) \\
&\cdots \\
\dot{z}_n &= q_n(z) + \frac{p_n(z)}{a(z)}\Big(-b(z)+v\Big)
\end{aligned}
\right\}\text{zero dynamics}
$$

where

$$a(z) = L_g L_f^{r-1} h(\Phi^{-1}(z)) \tag{16}$$

$$b(z) = L_f^r h(\Phi^{-1}(z)) \tag{17}$$

$$q_i(z) = L_f \phi_i(\Phi^{-1}(z)) \ , \quad i = \overline{r+1, n} \tag{18}$$

$$p_i(z) = L_g \phi_i(\Phi^{-1}(z)) \ , \quad i = \overline{r+1, n} \tag{19}$$

$$\Phi(z) = [\phi_i(z)]_{\text{col}}^{i=\overline{i,n}} \tag{20}$$

3 Numerical examples

3.1 Continuous fermentation process

The input affine model of the centered system is:

$$f(x) = \begin{bmatrix} \mu(S_0+x_2)(X_0+x_1) - \frac{F_0}{V}(X_0+x_1) \\ -\frac{F_0}{V}(S_0+x_2-S_F) - \frac{\mu(S_0+x_2)}{Y}(X_0+x_1) \end{bmatrix} \tag{21}$$

$$g(x) = -\frac{1}{V}\begin{bmatrix} X_0+x_1 \\ S_0+x_2-S_F \end{bmatrix} \tag{22}$$

$$\dot{x} = f(x) + g(x)u \tag{23}$$

Model parameters and their numerical values:

$$V = 4 \tag{24}$$
$$S_F = 10 \tag{25}$$
$$Y = 0.5 \tag{26}$$
$$\mu_{max} = 1 \tag{27}$$
$$K_1 = 0.03 \tag{28}$$
$$K_2 = 0.5 \tag{29}$$

Numerical values of the optimal operating point:

$$X_0 = 4.8907 \tag{30}$$
$$S_0 = 0.21866 \tag{31}$$
$$F_0 = 3.2089 \text{ optimal input flow rate} \tag{32}$$

Auxiliary variables:

$$c_0 = K_2 S_0^2 + S_0 + K_1 \tag{33}$$
$$c_1 = 2K_2 S_0 + 1 \tag{34}$$
$$c_2 = K_2 \tag{35}$$
$$q(x_2) = c_2 x_2^2 + c_1 x_2 + c_0 \text{ (denominator of the rational terms)} \tag{36}$$

The system's equation can be written in the form:

$$A(x) = \begin{bmatrix} -\frac{F_0 c_2 x_2^2 + F_0 c_1 x_2}{V q(x_2)} & \frac{\mu_{max}(X_0+x_1)}{q(x_2)} - \frac{F_0 X_0 (c_1 + c_2 x_2)}{V q(x_2)} \\ -\frac{S_0 \mu_{max}}{Y q(x_2)} & -\frac{F_0}{V} - \frac{\mu_{max}(X_0+x_1)}{Y q(x_2)} - \frac{F_0 (S_0 - S_F)(c_1 + c_2 x_2)}{V q(x_2)} \end{bmatrix}$$
$$B(x) = g(x)$$
$$f(x) = A(x)x + B(x)u \tag{37}$$

During the optimization we introduced the following constraints:

- P is a constant matrix (Q is constant as well and $\dot{Q}$ is zero)
- $N(x)$ and $M(x)$ are affine matrix functions in the elements of $\pi^{(N)}$ and $\pi^{(M)}$, respectively, where $\pi^{(N)}$ and $\pi^{(M)}$ contains monomials of x

$$\begin{aligned} N(x) &= C(x)Q = \sum_{i=1}^{p^{(N)}} N_i \, \pi_i^{(N)} \\ M(x) &= F(x)Q = \sum_{i=1}^{p^{(M)}} M_i \, \pi_i^{(M)} \end{aligned} \tag{38}$$

- G is constant

During the numerical computations, we used:

$$\pi^{(N)} = \begin{bmatrix} 1 & x_1 & x_2 \end{bmatrix}^T \tag{39}$$

$$\pi^{(M)} = \begin{bmatrix} 1 \end{bmatrix}^T \quad \text{i.e. } M \text{ is constant} \Rightarrow F = MP \text{ is constant} \tag{40}$$

In this special case, the obtained $h(x) = C(x)x = N(x)Px$ will be a second order polynomial function.

The grid, on which the optimization was done is:

$$x = (x_1, x_2) \in \mathfrak{L}^{15}_{[-1\,,\,1]} \times \mathfrak{L}^{15}_{[-0.1\,,\,0.2]} \tag{41}$$

3.2 Calculating the zero dynamics

The nonlinear coordinates transformation is defined as follows

$$z = \Phi(x) = \begin{bmatrix} h(x) \\ \lambda(x) \end{bmatrix} \tag{42}$$

where $\lambda(x)$ is defined in [biorektor] as

$$\lambda(x) = \frac{V(S_0 + x_2 - S_F)}{X_0 + x_1}, \quad \lambda(0) = -8 \tag{43}$$

3.3 Notations

$$\mathfrak{L}^{n}_{[a\,,\,b]} = \left\{ a + \frac{k-1}{n-1}(b-a),\ k = \overline{1,n} \right\} = \texttt{linspace}(\texttt{a},\texttt{b},\texttt{n}) \tag{44}$$

4 Zero state detectability (Inverted pendulum)

Kérdés:

$$\forall C(x) = x^T C_1 + C_0 \text{ -ra } ?\exists x_1^* \neq 0 \text{ ú.h. } C(x)x^* = 0 \tag{45}$$

$$\begin{bmatrix} C_0 \\ C_1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ \hline a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \\ a_{51} & a_{52} & a_{53} & a_{54} \end{bmatrix} \quad , \quad x^* = \begin{bmatrix} x_1^* \\ 0 \\ \pi \\ 0 \end{bmatrix} \tag{46}$$

$$\text{I. Konstans eset} \tag{47}$$

$$C = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \end{bmatrix} \tag{48}$$

$$Cx^* = 0 \Longrightarrow a_{11}x_1^* + a_{13}\pi = 0 \Longrightarrow x_1^* = -\frac{\pi a_{13}}{a_{11}} \tag{49}$$

$$\text{II. Lineáris } C(x) = x^T C_1 + C_0 \text{ eset} \tag{50}$$

$$C(x) = \begin{bmatrix} a_{11} + a_{21}x_1 + a_{31}x_2 + a_{41}x_3 + a_{51}x_4 \\ a_{12} + a_{22}x_1 + a_{32}x_2 + a_{42}x_3 + a_{52}x_4 \\ a_{13} + a_{23}x_1 + a_{33}x_2 + a_{43}x_3 + a_{53}x_4 \\ a_{14} + a_{24}x_1 + a_{34}x_2 + a_{44}x_3 + a_{54}x_4 \end{bmatrix}^T \tag{51}$$

$$C(x^*)x^* = a_{21}x^2 + ((a_{23} + a_{41})\pi + a_{11})x + a_{43}\pi^2 + a_{13}\pi = 0 \tag{52}$$

Diszkrimináns:

$$\begin{aligned} \Delta = B^2 - 4AC = {a_{11}}^2 + 2\pi a_{11}a_{23} + 2\pi a_{11}a_{41} + \pi^2 {a_{23}}^2 \\ + 2\pi^2 a_{23}a_{41} + \pi^2 {a_{41}}^2 - 4\pi a_{13}a_{21} - 4\pi^2 a_{21}a_{43} \end{aligned} \tag{53}$$

Egyenlet megoldása:

$$x^* = \frac{-B \pm \sqrt{\Delta}}{2A} = -\frac{a_{11} + \pi a_{23} + \pi a_{41} \pm \sqrt{\Delta}}{2a_{21}} \tag{54}$$

Következtetés: van olyan $C(x)$ lineáris mátrix, amelyre $C(x^*)x^* \neq 0$, bármely x_1^* esetén. Ez volt a kérdés?

5 Inverz inga

$$a(z) = \Big[L_f h(x)\Big]_{x=\Phi^{-1}(z)} \tag{55}$$