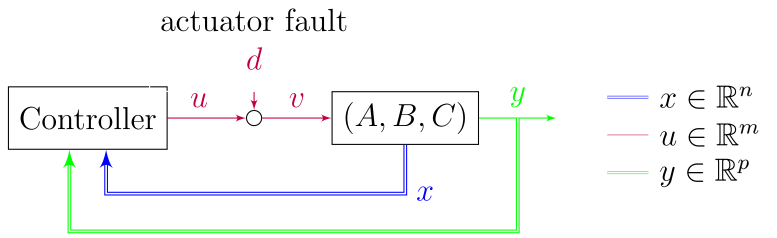
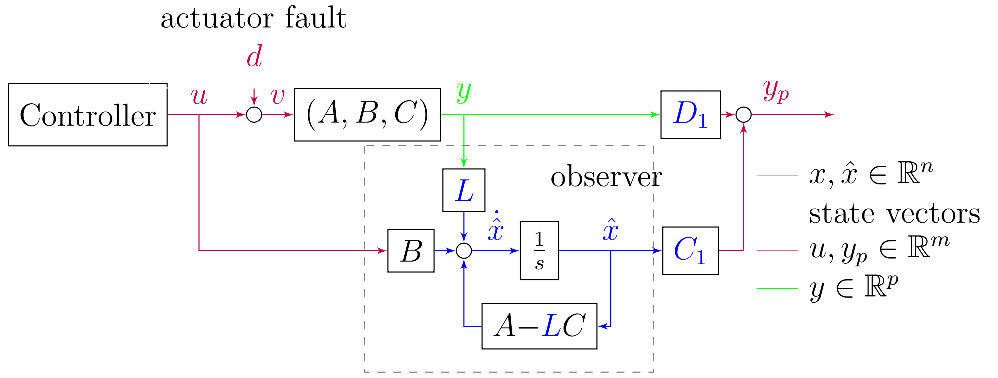
1 MIMO, $u \to y_p$ passivization, feedback equivalence with a passive system



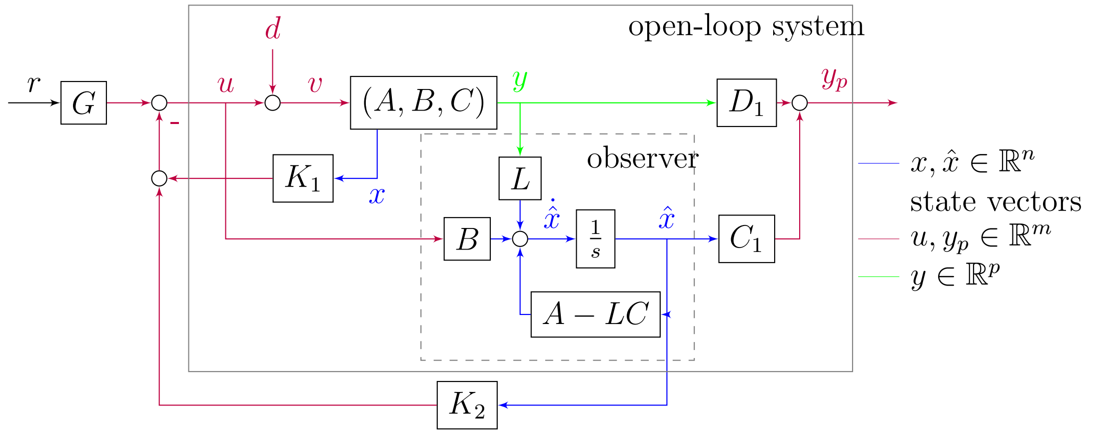
Having an LTI MIMO system, which is somehow fed back through a controller (either tuned by output vector or by the full state vector). The actuator is assumed to be faulty. We intend to detect its fault using system inversion.]. However, the system is not invertible, since its zeros are unstable and/or its vector relative degree (v.r.d.) is more than 1, therefore, we augment the system with an additional (linear) dynamics, which is tuned by the system's output y and the designed control input u, hence its resemblance to an observer.



Proposition 1. (Vector relative degree) *LTI MIMO*

If matrix B is full column rank, $\exists C_1$ such that the system $u \to y_p$ will have r.d. $\{1, .., 1\}$.

The goal is to chose matrices L, C_1, D_1 such that the system $u \to y_p$ be feedback equivalent to a passive system and the v.r.d. of $u \to y_p$ be 1.



1.1 Numerical example – MIMO

Having an LTI system with matrices:

$$A = \begin{pmatrix} 6.1 & 1.4 & -0.33 & -1.8 & 0.88 & -0.88 \\ -3.2 & 0.5 & 0.27 & 0.14 & -0.068 & 0.068 \\ 1.4 & -1.5 & 0.58 & 0.14 & -0.07 & 0.07 \\ -0.76 & -1.2 & -0.13 & 0.85 & 1.1 & -1.1 \\ 0.38 & 0.61 & 0.067 & 2.7 & 3.6 & -1.6 \\ -0.38 & -0.61 & -0.067 & 0.59 & 1.7 & 0.3 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \\ -1.6 & 1 \\ 0.82 & 2 \\ -0.82 & 0 \end{pmatrix} \tag{1}$$

$$C = \begin{pmatrix} -0.15 & -0.24 & -0.026 & -0.2 & 0.6 & -0.6 \\ 0.39 & 0.17 & -1 & 0.58 & 0.21 & -1.2 \end{pmatrix}$$

Its transfer function and its zeros are:

$$H(s) = \begin{pmatrix} \dfrac{s-1}{(s+1)(s-2)} & \dfrac{1}{s-3} \\ \dfrac{(s+2)(s-7)}{(s-5)(s-1)(s-2)} & \dfrac{s-6}{(s-2)(s-3)} \end{pmatrix} \tag{2}$$

$$\texttt{tzero(H)} = \begin{pmatrix} 7.2854 & 0.0214 & 3 & 2 & 1.8361 \end{pmatrix}$$

This system does not have a relative degree $\{1, 1\}$, since

$$\dot{y} = CAx + CBu, \quad CB = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \tag{3}$$

2 $X = \begin{pmatrix} x \\ \hat{x} \end{pmatrix}$, $L = \texttt{place}$, partially linearised by $N := Q\tilde{C}^T$

$$\text{OLS:} \quad \begin{cases} \dot{x} = Ax + Bu \\ \dot{\hat{x}} = LCx + (A - LC)\hat{x} + Bu \\ y_p = D_1 Cx + C_1 \hat{x} \end{cases} \Rightarrow \begin{cases} \tilde{A}_o = \begin{pmatrix} A & 0 \\ LC & A - LC \end{pmatrix}, \quad \tilde{B}_o = \begin{pmatrix} B \\ B \end{pmatrix} \\ \tilde{C}_o = \begin{pmatrix} D_1 C & C_1 \end{pmatrix} \end{cases} \tag{4}$$

$$\text{CLS:} \quad u = -KX + Gu \quad \Rightarrow \begin{cases} \tilde{A} = \begin{pmatrix} A & 0 \\ LC & A - LC \end{pmatrix} - BK, \quad \tilde{B} = \tilde{B}_o G \\ \tilde{C} = \begin{pmatrix} D_1 C & C_1 \end{pmatrix} \end{cases} \tag{5}$$

$$\dot{V}(X) \le r^T y_p + y_p^T r - y_p^T W y_p, \quad \text{where } W \succ 0, \; V(X) = X^T P X, \; P = P^T \succ 0 \tag{6}$$

$$\begin{pmatrix} X \\ r \end{pmatrix}^T \begin{pmatrix} \tilde{A}^T P + P\tilde{A} & P\tilde{B} \\ \tilde{B}^T P & 0 \end{pmatrix} \begin{pmatrix} X \\ r \end{pmatrix} \le \begin{pmatrix} X \\ r \end{pmatrix}^T \begin{pmatrix} -\tilde{C}^T W \tilde{C} & \tilde{C}^T \\ \tilde{C} & 0 \end{pmatrix} \begin{pmatrix} X \\ r \end{pmatrix} \tag{7}$$

Equivalently: (it could be **only** negative **semi**-definite)

$$M_0 = \begin{pmatrix} \tilde{A}^T P + P\tilde{A} + \tilde{C}^T W \tilde{C} & P\tilde{B} - \tilde{C}^T \\ \tilde{B}^T P - \tilde{C} & 0 \end{pmatrix} \preceq 0, \quad \text{but } \not\prec 0 \tag{8}$$

Using Schur's complement lemma: (it could be **only** negative **semi**-definite)

$$M_1 = \begin{pmatrix} \tilde{A}^T P + P\tilde{A} & P\tilde{B} - \tilde{C}^T & \tilde{C}^T \\ \tilde{B}^T P - \tilde{C} & 0 & 0 \\ \tilde{C} & 0 & -W^{-1} \end{pmatrix} \preceq 0, \quad \text{but } \not\prec 0 \quad \text{(BMI)} \tag{9}$$

This condition will remain bilinear since we co-design L, K but can efficiently be solved though.

$$\widehat{M}_1 := \begin{pmatrix} Q & & \\ & I & \\ & & I \end{pmatrix} M_1 \begin{pmatrix} Q & & \\ & I & \\ & & I \end{pmatrix} = \begin{pmatrix} Q\tilde{A}^T + \tilde{A}Q & \tilde{B} - Q\tilde{C}^T & Q\tilde{C}^T \\ \tilde{B}^T - \tilde{C}Q & 0 & 0 \\ \tilde{C}Q & 0 & -W^{-1} \end{pmatrix} \preceq 0 \tag{10}$$

The free decision variables of the bilinear problem are: Q, K, G, $N = Q\tilde{C}^T$. The bilinear problem is:

$$\widehat{M}_2 = \begin{pmatrix} Q\tilde{A}^T + \tilde{A}Q & \tilde{B} - N & N \\ \tilde{B}^T - N^T & 0 & 0 \\ N^T & 0 & -W^{-1} \end{pmatrix} \preceq 0 \quad \text{(BMI)} \tag{11}$$

$$Q \succeq 0, \quad G \neq 0 \tag{12}$$

After this:

$$\tilde{C} = (Q^{-1}N)^T = \begin{pmatrix} \tilde{C}_1 & \tilde{C}_2 \end{pmatrix} \overset{!}{=} \begin{pmatrix} D_1 C & C_1 \end{pmatrix}$$

$$\text{thus:} \quad \begin{cases} \tilde{C}_1 = D_1 C \text{ (overdetermined)} \quad \Rightarrow \quad D_1 = \tilde{C}_1 C^T \left(CC^T\right)^{-1} \text{ if } \mathrm{Im}(C^T) \overset{!!!}{\subseteq} \mathrm{Im}(\tilde{C}_1^T) \\ \tilde{C}_2 = C_1 \end{cases} \tag{13}$$

2.1 Numerical results

$$L^T = \begin{pmatrix} -9.45 & -82.2 & 206 & 57.3 & -193 & -199 \\ -85.7 & 58.7 & -50.2 & -0.158 & 16 & 16 \end{pmatrix}$$

$$C_1 = \begin{pmatrix} 0.0792 & -0.0938 & 0.143 & 0.46 & 0.108 & 0.751 \\ 1.42 & 1.34 & -0.379 & 2.09 & -1.09 & 3.34 \end{pmatrix}, \quad D_1 = \begin{pmatrix} 0.702 & -0.293 \\ 5.25 & -2.07 \end{pmatrix}$$

$$K_1 = \begin{pmatrix} 205 & -4.62 & 55.1 & -96.9 & 14.5 & -89.5 \\ 52.8 & -4.71 & -32.7 & 9.28 & 151 & 173 \end{pmatrix}$$

$$K_2 = \begin{pmatrix} -38.2 & -97.5 & 25.2 & 70.4 & -7.57 & 65.7 \\ 7.32 & -1.27 & 38 & 8.68 & 81.9 & 89.1 \end{pmatrix}, \quad G = \begin{pmatrix} 13.2 & 1.48 \\ 1.48 & 16 \end{pmatrix}$$

Problem. The closed loop system is indeed minimum phase, however, the open-loop system is NOT minimum phase, since D_1 does not exists such that $D_1 C = \tilde{C}_1$ (overdetermined).

If we can change C (i.e. we know the hole state vector), it may work.

3 $X = \begin{pmatrix} x \\ \hat{x} \end{pmatrix}$, $L =$ `sdpvar`, highly bilinear

$$\text{OLS:} \quad \begin{cases} \dot{x} = Ax + Bu \\ \dot{\hat{x}} = LCx + (A - LC)\hat{x} + Bu \\ y_p = D_1 Cx + C_1 \hat{x} \end{cases} \Rightarrow \quad \begin{cases} \tilde{A}_o = \begin{pmatrix} A & 0 \\ LC & A - LC \end{pmatrix}, \quad \tilde{B}_o = \begin{pmatrix} B \\ B \end{pmatrix} \\ \tilde{C}_o = \begin{pmatrix} D_1 C & C_1 \end{pmatrix} \end{cases} \tag{14}$$

$$\text{CLS:} \quad u = -KX + Gu \quad \Rightarrow \quad \begin{cases} \tilde{A} = \begin{pmatrix} A & 0 \\ LC & A - LC \end{pmatrix} - BK, \quad \tilde{B} = \tilde{B}_o G \\ \tilde{C} = \begin{pmatrix} D_1 C & C_1 \end{pmatrix} \end{cases} \tag{15}$$

$$\dot{V}(X) \le r^T y_p + y_p^T r - y_p^T W y_p \ , \quad \text{where } W \succ 0, \ V(X) = X^T P X, \ P = P^T \succ 0 \tag{16}$$

$$\begin{pmatrix} X \\ r \end{pmatrix}^T \begin{pmatrix} \tilde{A}^T P + P\tilde{A} & P\tilde{B} \\ \tilde{B}^T P & 0 \end{pmatrix} \begin{pmatrix} X \\ r \end{pmatrix} \le \begin{pmatrix} X \\ r \end{pmatrix}^T \begin{pmatrix} -\tilde{C}^T W \tilde{C} & \tilde{C}^T \\ \tilde{C} & 0 \end{pmatrix} \begin{pmatrix} X \\ r \end{pmatrix} \tag{17}$$

Equivalently: (it could be **only** negative **semi**-definite)

$$M_0 = \begin{pmatrix} \tilde{A}^T P + P\tilde{A} + \tilde{C}^T W \tilde{C} & P\tilde{B} - \tilde{C}^T \\ \tilde{B}^T P - \tilde{C} & 0 \end{pmatrix} \preceq 0, \quad \text{but } \not\prec 0 \tag{18}$$

Using Schur's complement lemma: (it could be **only** negative **semi**-definite)

$$M_1 = \begin{pmatrix} \tilde{A}^T P + P\tilde{A} & P\tilde{B} - \tilde{C}^T & \tilde{C}^T \\ \tilde{B}^T P - \tilde{C} & 0 & 0 \\ \tilde{C} & 0 & -W^{-1} \end{pmatrix} \preceq 0, \quad \text{but } \not\prec 0 \quad \text{(BMI)} \tag{19}$$

$$Q \succeq 0, \quad G \ne 0 \tag{20}$$

The free decision variables of the bilinear problem are: P, C_1, D_1, L, K, G.

3.1 Numerical results

$$L^T = \begin{pmatrix} -9.45 & -82.2 & 206 & 57.3 & -193 & -199 \\ -85.7 & 58.7 & -50.2 & -0.158 & 16 & 16 \end{pmatrix}$$

$$C_1 = \begin{pmatrix} 0.0792 & -0.0938 & 0.143 & 0.46 & 0.108 & 0.751 \\ 1.42 & 1.34 & -0.379 & 2.09 & -1.09 & 3.34 \end{pmatrix}, \quad D_1 = \begin{pmatrix} 0.702 & -0.293 \\ 5.25 & -2.07 \end{pmatrix}$$

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Problem. Solution not found.

4 $X = \begin{pmatrix} x \\ \hat{x} \end{pmatrix}$, $L =$ `place/sdpvar`, $G = I_m$, $D_1, C_1 =$ `sdpvar`

$$\text{OLS:} \quad \begin{cases} \dot{x} = Ax + Bu \\ \dot{\hat{x}} = LCx + (A - LC)\hat{x} + Bu \\ y_p = D_1 Cx + C_1 \hat{x} \end{cases} \Rightarrow \quad \begin{cases} \tilde{A}_o = \begin{pmatrix} A & 0 \\ LC & A - LC \end{pmatrix}, \quad \tilde{B}_o = \begin{pmatrix} B \\ B \end{pmatrix} \\ \tilde{C}_o = \begin{pmatrix} D_1 C & C_1 \end{pmatrix} \end{cases} \tag{21}$$

$$\text{CLS:} \quad u = -KX + Gu \quad \Rightarrow \quad \begin{cases} \tilde{A} = \begin{pmatrix} A & 0 \\ LC & A - LC \end{pmatrix} - BK, \quad \tilde{B} = \tilde{B}_o G \\ \tilde{C} = \begin{pmatrix} D_1 C & C_1 \end{pmatrix} \end{cases} \tag{22}$$

$$\dot{V}(X) \le r^T y_p + y_p^T r - y_p^T W y_p \ , \quad \text{where } W \succ 0, \ V(X) = X^T P X, \ P = P^T \succ 0 \tag{23}$$

$$\begin{pmatrix} X \\ r \end{pmatrix}^T \begin{pmatrix} \tilde{A}^T P + P\tilde{A} & P\tilde{B} \\ \tilde{B}^T P & 0 \end{pmatrix} \begin{pmatrix} X \\ r \end{pmatrix} \le \begin{pmatrix} X \\ r \end{pmatrix}^T \begin{pmatrix} -\tilde{C}^T W \tilde{C} & \tilde{C}^T \\ \tilde{C} & 0 \end{pmatrix} \begin{pmatrix} X \\ r \end{pmatrix} \tag{24}$$

Equivalently: (it could be **only** negative **semi**-definite)

$$M_0 = \begin{pmatrix} \tilde{A}^T P + P\tilde{A} + \tilde{C}^T W \tilde{C} & P\tilde{B} - \tilde{C}^T \\ \tilde{B}^T P - \tilde{C} & 0 \end{pmatrix} \preceq 0, \quad \text{but } \not\prec 0 \tag{25}$$

Using Schur's complement lemma: (it could be **only** negative **semi**-definite)

$$M_1 = \begin{pmatrix} \tilde{A}^T P + P\tilde{A} & P\tilde{B} - \tilde{C}^T & \tilde{C}^T \\ \tilde{B}^T P - \tilde{C} & 0 & 0 \\ \tilde{C} & 0 & -W^{-1} \end{pmatrix} \preceq 0, \quad \text{but } \not\prec 0 \quad \text{(BMI)} \tag{26}$$

$$P \succeq 0 \tag{27}$$

The free decision variables of the bilinear problem are: P, C_1, D_1, L, K,

Problem. Solution not found.

5 Separability principle

D_1 should be full rank!

$$\text{OLS:} \quad \begin{cases} \dot{x} = Ax + Bu \\ y_p = (D_1 C + C_1)x \end{cases} \quad u = -Kx + Gu \quad \Rightarrow \quad \text{CLS:} \quad \begin{cases} \tilde{A} = A - BK, \quad \tilde{B} = BG \\ \tilde{C} = D_1 C + C_1 \end{cases} \tag{28}$$

$$\dot{V}(x) \le r^T y_p + y_p^T r - y_p^T W y_p \ , \quad \text{where } W \succ 0, \ V(x) = x^T P x, \ P = P^T \succ 0 \tag{29}$$

$$\begin{pmatrix} x \\ r \end{pmatrix}^T \begin{pmatrix} \tilde{A}^T P + P\tilde{A} & P\tilde{B} \\ \tilde{B}^T P & 0 \end{pmatrix} \begin{pmatrix} x \\ r \end{pmatrix} \le \begin{pmatrix} x \\ r \end{pmatrix}^T \begin{pmatrix} -\tilde{C}^T W \tilde{C} & \tilde{C}^T \\ \tilde{C} & 0 \end{pmatrix} \begin{pmatrix} x \\ r \end{pmatrix} \tag{30}$$

Equivalently: (it could be **only** negative **semi**-definite)

$$M_0 = \begin{pmatrix} \tilde{A}^T P + P\tilde{A} + \tilde{C}^T W \tilde{C} & P\tilde{B} - \tilde{C}^T \\ \tilde{B}^T P - \tilde{C} & 0 \end{pmatrix} \preceq 0, \quad \text{but } \not\prec 0 \tag{31}$$

Using Schur's complement lemma: (it could be **only** negative **semi**-definite)

$$M_1 = \begin{pmatrix} \tilde{A}^T P + P\tilde{A} & P\tilde{B} - \tilde{C}^T & \tilde{C}^T \\ \tilde{B}^T P - \tilde{C} & 0 & 0 \\ \tilde{C} & 0 & -W^{-1} \end{pmatrix} \preceq 0, \quad \text{but } \not\prec 0 \quad \text{(BMI)} \tag{32}$$

This condition will remain bilinear since we co-design L, K but can efficiently be solved though.

$$\widehat{M}_1 := \begin{pmatrix} Q & & \\ & I & \\ & & I \end{pmatrix} M_1 \begin{pmatrix} Q & & \\ & I & \\ & & I \end{pmatrix} = \begin{pmatrix} Q\tilde{A}^T + \tilde{A}Q & \tilde{B} - Q\tilde{C}^T & Q\tilde{C}^T \\ \tilde{B}^T - \tilde{C}Q & 0 & 0 \\ \tilde{C}Q & 0 & -W^{-1} \end{pmatrix} \preceq 0 \tag{33}$$

The free decision variables of the bilinear problem are: Q, K, G, $N = Q\tilde{C}^T$. The bilinear problem is:

$$\widehat{M}_2 = \begin{pmatrix} Q\tilde{A}^T + \tilde{A}Q & \tilde{B} - N & N \\ \tilde{B}^T - N^T & 0 & 0 \\ N^T & 0 & -W^{-1} \end{pmatrix} \preceq 0 \quad \text{(BMI)} \tag{34}$$

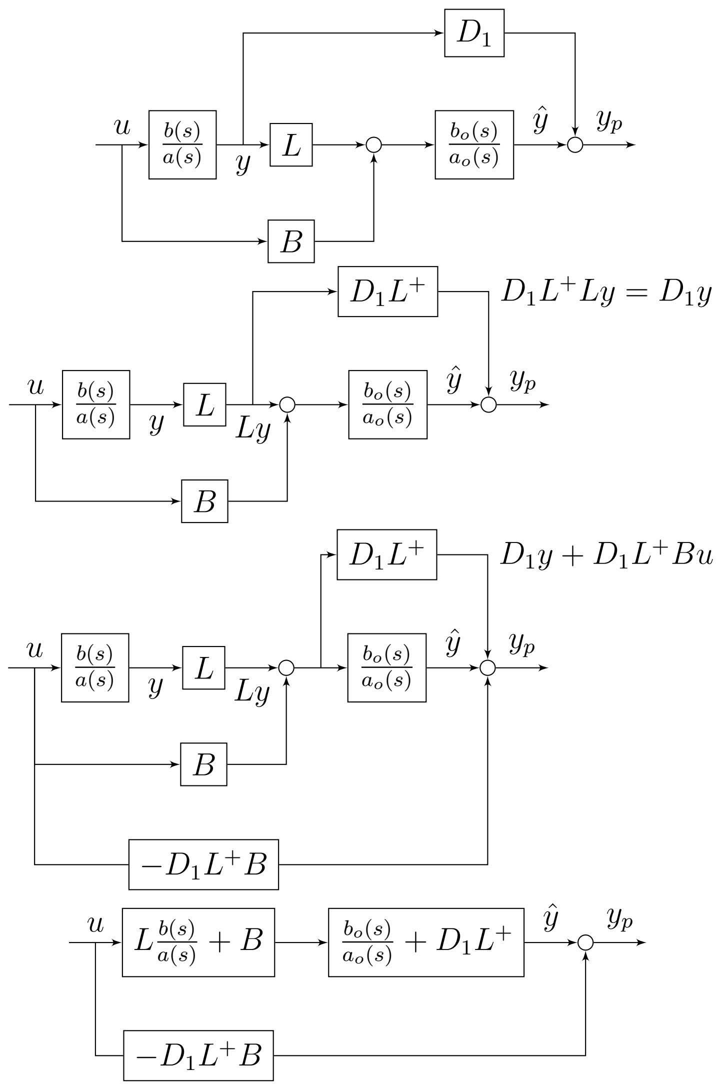
$$Q \succeq 0, \quad G \ne 0 \tag{35}$$

After this:

$$\tilde{C} = (Q^{-1}N)^T = \begin{pmatrix} \tilde{C}_1 & \tilde{C}_2 \end{pmatrix} \overset{!}{=} \begin{pmatrix} D_1 C & C_1 \end{pmatrix}$$

$$\text{thus:} \quad \begin{cases} \tilde{C}_1 = D_1 C \ \text{(overdetermined)} \quad \Rightarrow \quad D_1 = \tilde{C}_1 C^T \left(CC^T\right)^{-1} \text{ if } \mathrm{Im}(\mathrm{C}^T) \overset{!!!}{\subseteq} \mathrm{Im}(\tilde{\mathrm{C}}_1^T) \\ \tilde{C}_2 = C_1 \end{cases} \tag{36}$$

6 Megvalósíthatósági tanulmány – operátortartománybeli analízis



Ahol

$$\frac{b_o(s)}{a_o(s)} = C_1\left(sI - A + LC\right)^{-1} \tag{37}$$

Szerintem ez MIMO esetben is jó ($a(s)$ és $a_o(s)$ skalár, minden egyéb mátrix):

$$\begin{aligned}
y_p &= D_1y + \hat{y} = \frac{1}{a(s)}D_1b(s)u + \frac{1}{a_o(s)}b_o(s)\left(\frac{1}{a(s)}Lb(s) + B\right)u \\
G_e(s) &= \frac{1}{a(s)}D_1b(s) + \frac{1}{a_o(s)}b_o(s)\left(\frac{1}{a(s)}Lb(s) + B\right) \\
G_e(s) &= \frac{1}{a(s)a_o(s)}D_1b(s)a_o(s) + \frac{1}{a(s)a_o(s)}b_o(s)(Lb(s) + Ba(s)) \\
G_e(s) &= \frac{1}{a(s)a_o(s)}\Big(D_1b(s)a_o(s) + b_o(s)Lb(s) + b_o(s)Ba(s)\Big)
\end{aligned} \tag{38}$$

Tehát az új számláló:

$$b_e(s) = D_1b(s)a_o(s) + b_o(s)Lb(s) + b_o(s)Ba(s) \tag{39}$$

A piros tagok lehetnek instabilak. Kék: megválasztható szabad constans. Mivel $a_o(s)$ skalár:

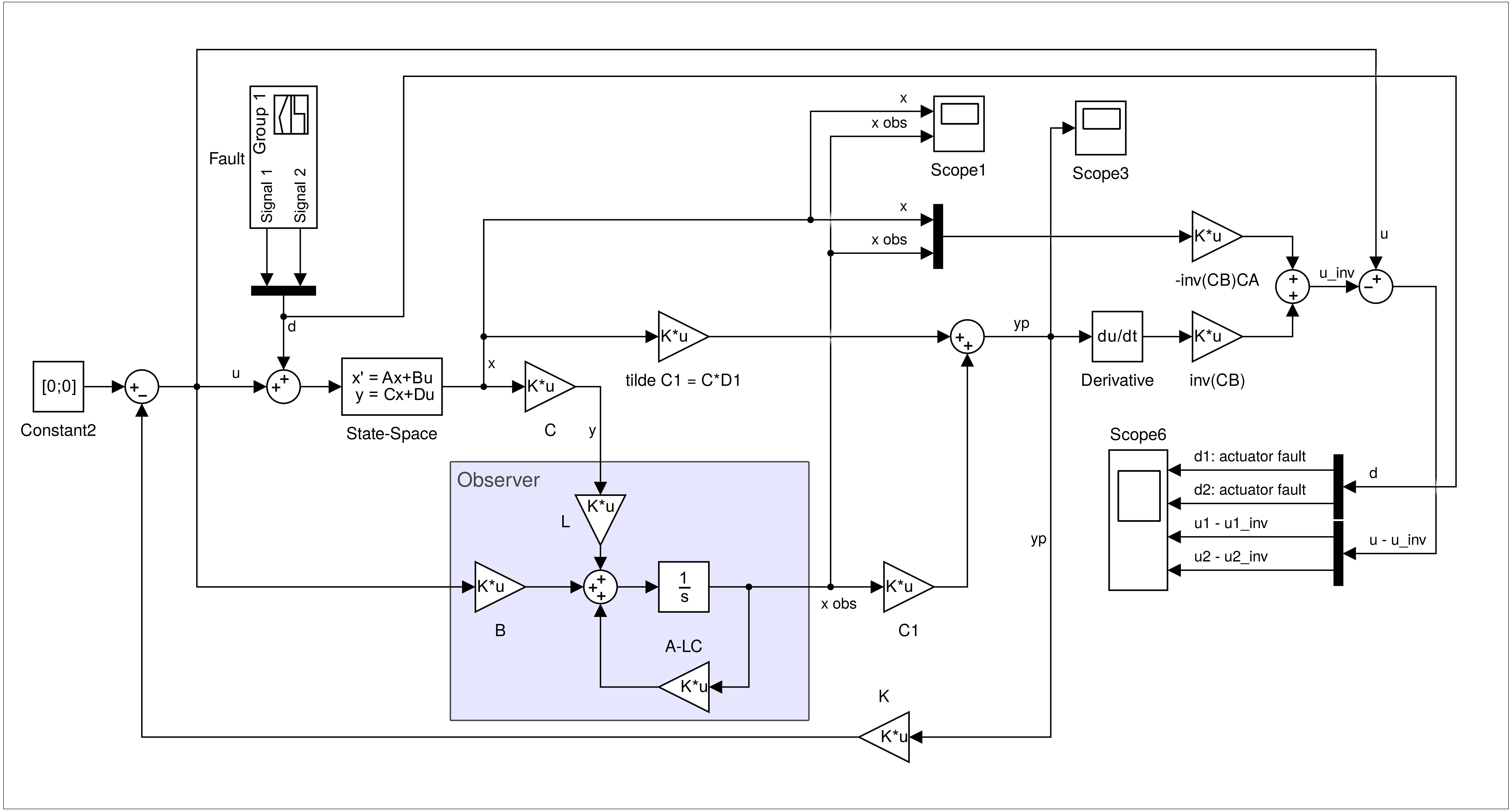
$$b_e(s) = \Big(a_o(s)D_1 + b_o(s)L\Big)b(s) + b_o(s)Ba(s) \leftarrow \text{ legyen stabil} \tag{40}$$

Numerikus példával illusztrálom, hogy ez valóban lehetséges:

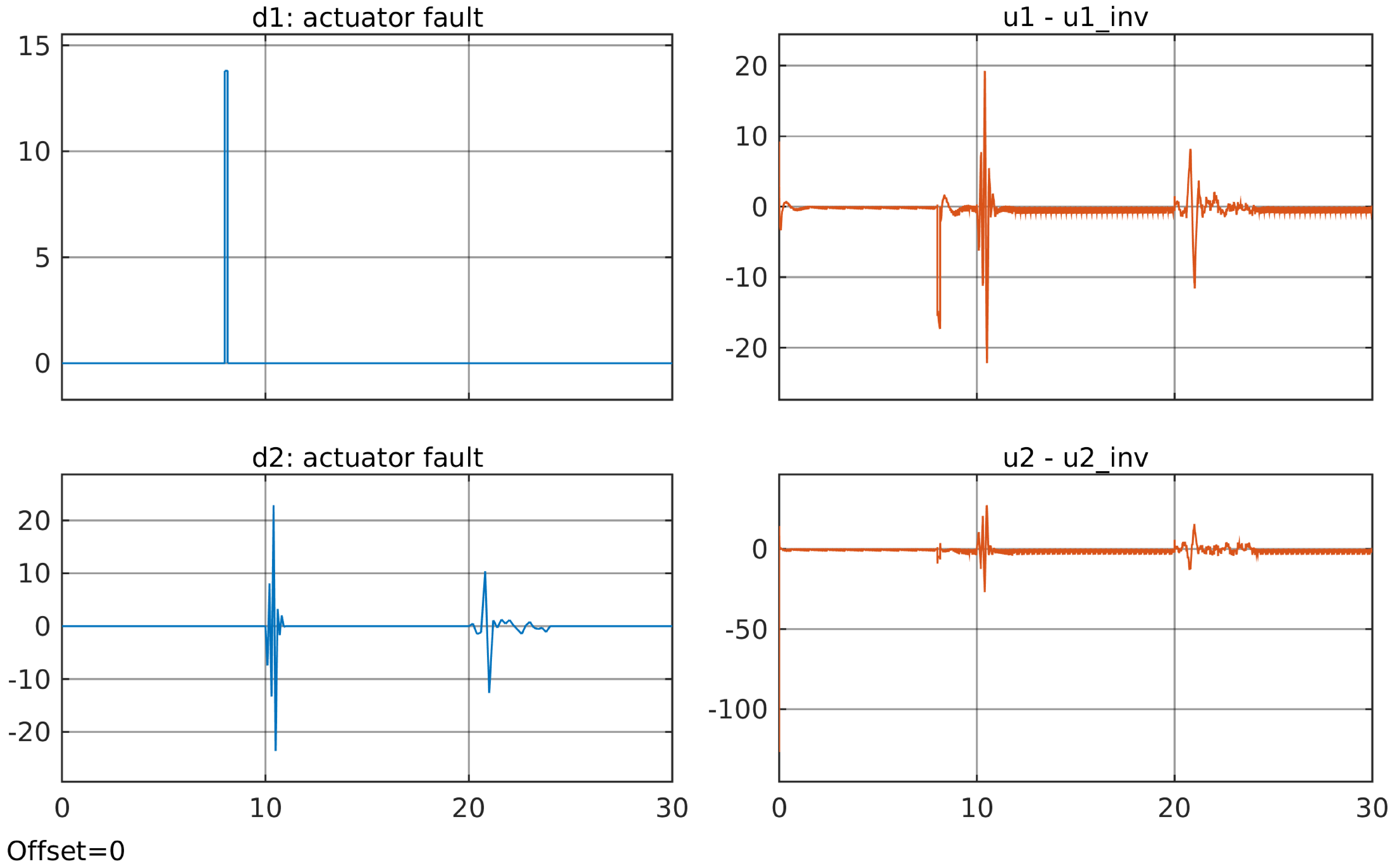
$$\begin{aligned}
&G(s) = \frac{b(s)}{a(s)} = \frac{s-5}{s^2-3s+2}, \quad A = \begin{pmatrix}3 & -2\\ 1 & 0\end{pmatrix}, \ B = \begin{pmatrix}2\\ 0\end{pmatrix}, \ C = \begin{pmatrix}0.5 & -2.5\end{pmatrix} \\
&\text{observer: } L \overset{\texttt{place}}{:=} \begin{pmatrix}-13\\ -5\end{pmatrix}, \ C_1 := \begin{pmatrix}1 & -1\end{pmatrix} \Rightarrow G_o(s) = \frac{b_o(s)}{a_o(s)} = \frac{1}{s^2+3s+2}\begin{pmatrix} s+9 & -s-25 \end{pmatrix} \\
&b_e(s) = (D_1+2)s^3 + (4-2D_1)s^2 + (-13D_1-2)s - (10D_1+4)\Big|_{D_1=-1} \\
&\quad = s^3 + 6s^2 + 11s + 6 = (s+3)(s+2)(s+1) \\
&\text{eredő átviteli függvény: } G_e(s) = \frac{s+3}{s^2-3s+2}
\end{aligned} \tag{41}$$

Másik példa (ami a relatív degreet is megjavítja):

$$\begin{aligned}
&G(s) = \frac{s-5}{(s-3)(s^2-3s+2)}, \quad L^T \overset{\texttt{place}}{:=} \begin{pmatrix}-193 & -212 & -52\end{pmatrix}, \ C_1 := \begin{pmatrix}1 & 1 & 1\end{pmatrix}, \ D_1 := 0.1 \\
&G_e(s) = \frac{s^2+4.1s+3.5}{s^3-6s^2+11s-6}
\end{aligned} \tag{42}$$



Fault
Group 1
Signal 1
Signal 2
d
[0;0]
Constant2
u
x' = Ax+Bu
y = Cx+Du
State-Space
x
K*u
C
y
tilde C1 = C*D1
x
x obs
Scope1
Scope3
-inv(CB)CA
u_inv
u
yp
du/dt
Derivative
inv(CB)
Scope6
d1: actuator fault
d2: actuator fault
u1 - u1_inv
u2 - u2_inv
d
u - u_inv
Observer
L
B
1
s
A-LC
x obs
C1
yp
K



d1: actuator fault
u1 - u1_inv
d2: actuator fault
u2 - u2_inv
15
10
5
0
20
10
0
-10
-20
-50
-100
0
10
20
30
Offset=0