Nonlinear Dynamical Systems (P-ITEEA-0037)

Lecture 2

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1. Pendulum

The equation of the damped pendulum is:

$$\ddot{y} + b\,\dot{y} + \sin(y) = \textit{external forces} \tag{1}$$

The (scaled) energy of the pendulum is:

$$\dot{y}^2/2 + (1 - \cos(y)) \tag{2}$$

The state-space representation of the pendulum system is

$$\begin{cases} x_1 = y \\ x_2 = \dot{y} \end{cases} \quad \Rightarrow \quad \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -bx_2 - \sin(x_1) + \textit{external forces} \end{cases} \tag{3}$$

Namely,

$$\boxed{\dot{x} = f(x)} \text{ where } x = \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right), \text{ and } f(x) = \begin{pmatrix} x_2 \\ -bx_2 - \sin(x_1) \end{pmatrix}. \tag{4}$$

Function $f : \mathbb{R}^n \to \mathbb{R}^n$ can be called the vector field of the ODE.

2. Van der Pol ocsillator

Van der Pol equation:

$$\ddot{y} - \mu(1 - y^2)\,\dot{y} + y = \textit{external forces} \tag{5}$$

State-space representation of the oscillator:

$$\begin{cases} x_1 = y \\ x_2 = \dot{y} \end{cases} \quad \Rightarrow \quad \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \mu(1 - x_1^2)\,x_2 - x_1 + \textit{external forces} \end{cases} \tag{6}$$

MATLAB can solve differential equation **numerically** of the form:

$$\boxed{\dot{x} = f(t, x)} \tag{7}$$

where the $f : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ is allowed to depend on time directly.

```
b = 0.1;  % <----- damping coefficient
f = @(t,x) [
    x(2)
    -b*x(2) - sin(x(1)) + 0.6*sin(0.6*t)
];

T = 10;     % <----- maximum time to simulate
x0 = [0;2];
[t,x] = ode45(f,[0,T],x0);

plot(t,x)           % <---- time plot  OR
plot(x(:,1),x(:,2)) % <---- phase diagram
```

2.1. Task

1. Visualize the energy function (`surf`), the energy levels (`contour`), and the vector field of the pendulum ODE (`quiver`).

2. Find an initial condition for (3) using Bolzano shooting, such that the (numerical) solution approaches a saddle point as the time goes to infinity.

   Alternatively: find the velocity necessary to flip the pendulum upward from the stable equilibrium point such that the pendulum does NOT swing over, but remains in the unstable equilibrium point (at least for a while).

   – use different damping coefficients –

3. Visualize the domain, in which an initial condition will lead to the trivial equilibrium point $(x_1 = 0, x_2 = 0)$.

   – use different damping coefficients –

4. Analyze the dynamical properties of the following differential equations:

   (a) $\ddot{y} + y = 0$,

   (b) $\ddot{y} + b\dot{y} + y = 0$,

   (c) $\ddot{y} + y = \cos(\omega t)$,

   (d) $\ddot{y} + b\dot{y} + y = \cos(\omega t)$,

   (e) $\ddot{y} + \sin(y) = 0$,

   (f) $\ddot{y} + b\dot{y} + \sin(y) = 0$,

   (g) $\ddot{y} + \sin(y) = \cos(\omega t)$,

   (h) $\ddot{y} + b\dot{y} + \sin(y) = \cos(\omega t)$,

   – use different damping coefficients –

5. Analyse the limit cycle of the Van der Pol oscillator for different parameters $\mu \in (0, \infty)$.

6. Find the slow and fast sections in the solutions of the Van der Pol equation (6).

7. Solve the ODEs with multiple initial conditions $(x_1(0), x_2(0)) \in \mathbb{R}^2$ from a grid in the state-space ($\mathbb{R}^2$). Plot the phase diagrams of all computed solutions.