Nonlinear Dynamical Systems (P-ITEEA-0037)

Lecture 1

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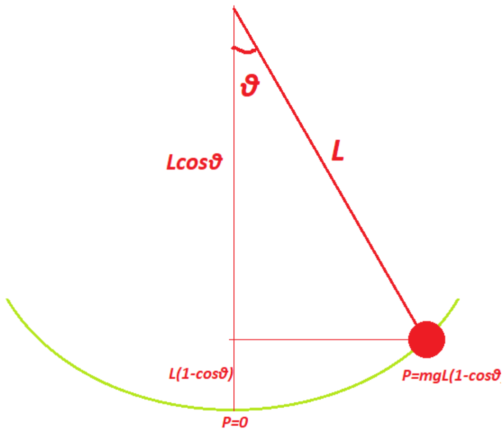
1. Pendulum

1.1. Energy



Kinetik energy of pendulum: $T = \frac{mv^2}{2} = \frac{mL^2\dot{\vartheta}^2}{2}$.

Potential energy of pendulum: $V = mgL(1 - \cos(\vartheta))$.

Lagrangian function of the system:

$$\mathcal{L}(\vartheta, \dot{\vartheta}) = T - V = \frac{mL^2\dot{\vartheta}^2}{2} - mgL(1 - \cos(\vartheta)) \tag{1}$$

The energy of the system is:

$$-\mathcal{L} + \dot{\vartheta} \cdot \frac{\partial \mathcal{L}}{\partial \dot{\vartheta}} = \frac{mL^2\dot{\vartheta}^2}{2} + mgL(1 - \cos(\vartheta)) \qquad \Big(= T + V\Big) \tag{2}$$

1.2. Equation of motion

Equation of motion using Euler-Lagrange equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\vartheta}}\right) - \frac{\partial \mathcal{L}}{\partial \vartheta} = M_{\text{ext}} \text{ (external torque and/or damping)} \tag{3}$$

The damping force is generally proportional to the velocity, but its direction is opposite to the direction of the velocity, therefore:

$$F_{\text{ext}} = -bv = -bL\dot{\vartheta} \tag{4}$$

In this case the external torque is

$$M_{\text{ext}} = L\, M_{\text{ext}} = -bL^2\dot{\vartheta} \tag{5}$$

Finally, the equation of motion is:

$$mL^2\ddot{\vartheta} + bL^2\dot{\vartheta} + mgL\sin(\vartheta) = 0 \tag{6}$$

Consider the following model simplifications:

$$\underbrace{m}_{1}\ddot{\vartheta} + b\dot{\vartheta} + \underbrace{\frac{mg}{L}}_{1}\sin(\vartheta) = 0 \tag{7}$$

Then, the simplified equation is:

$$\ddot{\vartheta} + b\,\dot{\vartheta} + \sin(\vartheta) = 0 \tag{8}$$

1.3. State-space representation

In order to solve the ordinary differential equation (ODE) (8) with MATLAB, we should write the ODE in the form of a **system of first order ODEs**.

Let us introduce the following variables:

$$\begin{cases} x_1 = \vartheta \\ x_2 = \dot{\vartheta} \end{cases} \Rightarrow \begin{cases} \dot{x}_1 = \dot{\vartheta} = x_2 \\ \dot{x}_2 = \ddot{\vartheta} = -b\dot{\vartheta} - \sin(\vartheta) = -bx_2 - \sin(x_1) \end{cases} \Rightarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -bx_2 - \sin(x_1) \end{cases} \tag{9}$$

If we introduce the vectorial (vector-valued) variable $x = \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right)$, then Eq. (9) can be written in the form:

$$\boxed{\dot{x} = f(x)} \text{ where } f(x) = f(x_1, x_2) = \begin{pmatrix} x_2 \\ -bx_2 - \sin(x_1) \end{pmatrix}. \tag{10}$$

Function $f : \mathbb{R}^n \to \mathbb{R}^n$ can be called the vector field of the ODE.

In fact, MATLAB can solve differential equation **numerically** of a more general form, namely:

$$\boxed{\dot{x} = f(t, x)} \tag{11}$$

where the $f : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ is allowed to depend on time directly.

1.4. Task

1. Visualize the energy function, the energy levels, and the vector field of the ODE (`surf`, `quiver`, `contour`).
2. Analyze the dynamical properties of the following differential equations:
   - (a) $\ddot{y} + y = 0$,
   - (b) $\ddot{y} + b\dot{y} + y = 0$,
   - (c) $\ddot{y} + y = \cos(\omega t)$,
   - (d) $\ddot{y} + b\dot{y} + y = \cos(\omega t)$,
   - (e) $\ddot{y} + \sin(y) = 0$,
   - (f) $\ddot{y} + b\dot{y} + \sin(y) = 0$,
   - (g) $\ddot{y} + \sin(y) = \cos(\omega t)$,
   - (h) $\ddot{y} + b\dot{y} + \sin(y) = \cos(\omega t)$,
3. Find an initial condition for $\ddot{y} + by + \sin(y) = 0$ using Bolzano shooting, such that the (numerical) solution approaches a saddle point as the time goes to infinity.