II./1.

- Melyik sorosattérben van: $\left.\begin{array}{l} a)\ x_n = \frac{1}{n} \\ b)\ y_n = (-1)^n + \frac{1}{n} \\ c)\ z_n = \left(1+\frac{1}{n}\right)^n \end{array}\right\}$ sorozat?
- Mennyi x_n, y_n, z_n normája?

$$\left[(a_n) \subset l_1 \rightarrow a_n \subset l_1 \subset \underset{p>1}{l^p} \subset l^\infty\right]$$

a) $\lim\limits_{n\to\infty} x_n = 0$, de $\sum_n \left|\frac{1}{n}\right| = \infty$, így $(x_n) \in c_0 \subset l^\infty$

norma: $\|x_n\|_\infty = \sup\limits_{n\in N}\{|x_n|\} = 1$

b) $y_n = \underbrace{(-1)^n}_{} + \frac{1}{n}$; $y_n \notin l_1$; c, ide $y \in l^\infty$ és $\|y\|_\infty = \frac{3}{2}$

-1 ha n páratlan
$+1$ ha n páros
$\left(0, \frac{3}{2}; -\frac{2}{3}; \frac{5}{4}; -\frac{4}{5}; \ldots\right)$

c) $\lim\limits_{n\to\infty} z_n = e$, így $z_n \in c$ és z_n korlátos is:

$$\left(0 < \boxed{\left(1+\frac{1}{n}\right)^n} = \sum_{k=0}^{n} \binom{n}{k}\left(\frac{1}{n}\right)^k = \sum_{k=0}^{n} \frac{n}{n}\underbrace{\frac{(n-1)\cdot(n-k+1)}{n \cdots n}}_{<1} \cdot \frac{1}{k!} \leq \sum_{k=0}^{n} \frac{1}{k!} < 1 + \sum_{k=0}^{n-1} \frac{1}{2^k} = 1 + \underbrace{\frac{1-\frac{1}{2^n}}{\frac{1}{2}}}_{\underbrace{1+2}_{3} - \frac{2}{2^n}} < \boxed{3}\right)$$

$\rightarrow$ Tehát $(z_n) \in l^\infty$

és $\|z_n\|_\infty = 3$.

$$\left((d)\ v_n = \frac{1}{2^n} \longrightarrow \sum_{n=0}^{\infty}\left(\frac{1}{2^n}\right)^2 = \sum_{n=0}^{\infty}\left(\frac{1}{2^2}\right)^n = \frac{1}{1-\frac{1}{4}} = \frac{4}{3};\right.$$

így $v_n \in l_2$ ($\cancel{\supset l_1}$); $\|v_n\|_\infty = \sup\limits_{n\in N}$ ~~max~~ $\{|v_n|\} = 1$)

II/2. $-\ x_n \equiv 1 \in c \subset l^\infty$, $\|x_n\|_\infty = 1$ ($c = \{x_n$ konvergens

$-\ x_n = (-1)^n \cdot \frac{1}{n}$; $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n}$ Leibniz sor, de

$\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n}\right| = \infty$, ezért $x_n \in c$

$\rightarrow\ x_n = \frac{1}{n^2}$; ismét: $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, így $x_n \subset l^\infty$, $\|x_n\|_\infty = 1$ ($x_n \subset c$)

$$l_1 = \{x_n;\ \sum_{n=1}^{\infty}|x_n| < \infty\}$$

3. Állítás: $l_1 \subset l_3$; $l_3 = \{x_n;\ \sum_{n=1}^{\infty}|x_n|^3 < \infty\}$

Biz: ha $(x_n) \subset l_1$; $\lim_{n\to\infty} x_n = 0$; (Analízis I.: ha $\sum_{n=1}^{\infty} a_n$ konvergens $\to \lim_{n\to\infty} a_n = 0$)

Így $\varepsilon = \frac{1}{2}\ \exists N: \underbrace{|x_n|}_{|x_n - x_0|} < \frac{1}{2}$; $\forall n \geq N$

$$(x_n) \in l_1 \to x_n \in l_3?\quad \sum_{n=1}^{M}|x_n|^3 \leq \underbrace{\sum_{n=1}^{N}|x_n|^3}_{\wedge\ \frac{1}{2}} + \underbrace{\sum_{n=N+1}^{M}|x_n|}_{M\to\infty \text{ esetén } < \infty}$$

(mert $|x_n| < \frac{1}{2}$)

Tehát ha $\sum_{n=1}^{\infty}|x_n| < \infty \to \sum_{n=1}^{\infty}|x_n|^3 < \infty$

4. Skalárszorzat $\to$ norma

$$\|x+y\|^2 = \langle x+y | x+y\rangle = \langle x|x\rangle + \langle y|y\rangle + 2\langle x|y\rangle \overset{\langle x|y\rangle}{=}$$

$$= \langle x|x\rangle + \langle y|y\rangle = \|x\|^2 + \|y\|^2 \qquad (\langle x|y\rangle = \langle y|x\rangle)$$

5. Ha $\|x\| = (\langle x|x\rangle)^{1/2}$

$$\underbrace{\langle x+y|x+y\rangle}_{\|x+y\|^2} + \underbrace{\langle x-y|x-y\rangle}_{\|x-y\|^2} = \langle x|x\rangle + \langle y|y\rangle +$$

$$2\langle x|y\rangle + \langle x|x\rangle + \underbrace{\langle -y|-y\rangle}_{(-1)^2\langle y|y\rangle} + \cancel{(-\langle x|y\rangle)} +$$

$$\cancel{(-\langle x|y\rangle)} = \langle x|x\rangle + \langle y|y\rangle = \|x\|^2 + \|y\|^2$$

6. $f(x) = x$; $g(x) = \sqrt{x}$; $C_{[0,1]} = \{f: [0,1] \to \mathbb{R};\ f \text{ folytonos}\}$

$$d_2(f(x), g(x)) = \|f(x) - g(x)\|_2 = \left(\int_0^1 (f(x) - g(x))^2 dx\right)^{1/2} =$$

$$= \left(\int_0^1 f^2(x)dx + \int_0^1 g^2(x)dx - 2\int_0^1 f(x)g(x)dx\right)^{1/2} =$$

$$\left(\cancel{= \left(\int_0^1 (x - \sqrt{x})^2 dx\right)^{1/2}} = \left(\int_0^1 x^2 dx + \int_0^1 \sqrt{x}\,dx - 2\int_0^1 x^{3/2} dx\right)^{1/2} \right.$$

$$= \left(\frac{x^3}{3}\Big|_0^1 + \cancel{\left(\frac{1}{2\sqrt{x}}\Big|_0^1\right)} + \frac{2x^{3/2}}{3}\Big|_0^1 - \cancel{\frac{4}{5}} x^{5/2}\Big|_0^1\right)^{1/2}$$

10.

$\|.\|_\infty$; $\lim\limits_{n\to\infty} \underbrace{\| f_n(x) - f(x) \|_\infty}_{\max\limits_{x\in[0,1]} |f_n(x) - f(x)| < \varepsilon \text{ ha } n \geq N \ (\forall x)} = \emptyset$

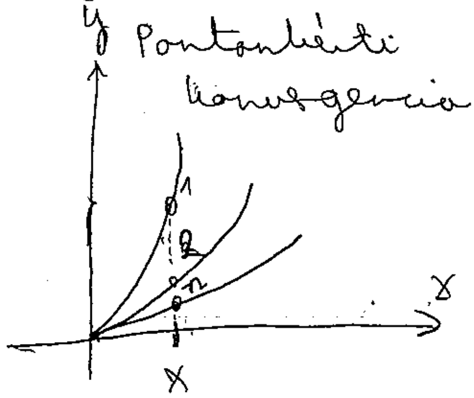
Mivel $\max |f_n(x) - f(x)| < \varepsilon$, ezért $\forall x$-re igaz a fenti állítás, azaz $\forall \varepsilon > 0$-hoz $\exists N = N(\varepsilon)$(*) hogy

$|f_n(x) - f(x)| < \varepsilon$; $n \geq N(\varepsilon)$(*) (N nem függ x-től).

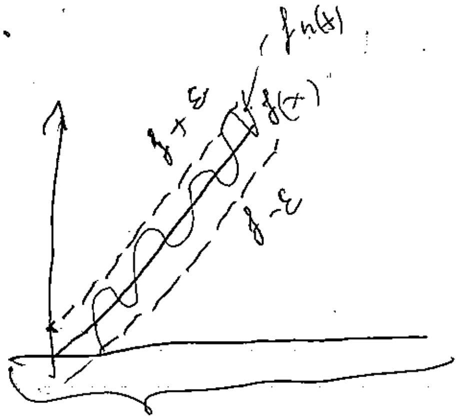
Tehát $f_n(x) \to f$ egyenletes konvergenciája esetén

$f_n(x) \to f$ $\|.\|_\infty$ szerint is.

M.:



Pontonkénti konvergencia



Egyenletes konvergencia

(*) N csak ε-tól függ ; x-től nem!

Állítás$_1$: $\bigcup_\alpha A_\alpha$ nyílt, ha A_α halmazok nyíltak.

Biz.: ha $x \in \bigcup_\alpha A_\alpha \to x \in A_\alpha$ valamely α-ra.
Mivel $x \in A_\alpha$, ezért x belső pont: $\exists r$; $B_r(x) \subset A_\alpha$.
Ezért $B_r(x) \subset \bigcup\limits_\alpha A_\alpha$

Állítás$_2$: $\bigcap\limits_{i=1}^{n} A_i$ nyílt, ha A_i halmazok nyíltak ($n < \infty$)

A_1, A_2: $A_1 \cap A_2 = \emptyset \to A_1 \cap A_2$ nyílt (üres halmaz nyílt)
$A_1 \cap A_2 \neq \emptyset \to \exists x \in A_1 \cap A_2 \xrightarrow[\exists r_1, r_2]{} B_{r_1}(x) \subset A_1$; $B_{r_2}(x) \subset A_2$
Legyen $r = \min(r_1, r_2) \longrightarrow B_r(x) \subset A_1 \cap A_2$; amiből
$x \in A_1 \cap A_2$, $A_i \{i = 1 \ldots n\} \to \exists r = \min\{r_1 \ldots r_n\}$; $B_r(x) \subset \bigcap_{i=1}^{n} A_i$

11./6.

$$\cancel{\left(= \left(\frac{1}{3}+\frac{2}{3}-\frac{4}{5}\right)^{1/2} = \frac{1}{\sqrt{5}}\right)}$$

$$= \left(\int_0^1 x^2\,dx + \int_0^1 x\,dx - 2\int_0^1 \overbrace{x\sqrt{x}}^{x^{3/2}}\,dx\right)^{1/2} = \left(\frac{x^3}{3}\Big|_0^1 + \frac{x^2}{2}\Big|_0^1 - \frac{4}{5}\right)^{1/2}$$

$$= \left(\underbrace{\frac{1}{3}+\frac{1}{2}}_{5/6} - \frac{4}{5}\right)^{1/2} = \frac{1}{\sqrt{30}}$$

$$\|f(x)-g(x)\|_\infty = \max_{x\in[0,1]}\left(|f(x)-g(x)|\right)$$

x maximum hely $\longrightarrow$ $\frac{d}{dx}\left(x-\sqrt{x}\right) = \phi$; $\frac{d}{dx}\left(x-\sqrt{x}\right) = 1-\frac{1}{2\sqrt{x}} = \phi$

$$\boxed{x = \frac{1}{4}}$$

maximum értéke : $\left|\frac{1}{4} - \sqrt{\frac{1}{4}}\right| = \boxed{\frac{1}{2}}$

11/8. (a) Áll: $\left(\overset{\mathbb{R}^n}{\cancel{C_{[0,1]}}}, \|.\|_\infty\right)$ nem származtatható skaláris szorzatból.

$$\|x+y\|^2 + \|x-y\|^2 \overset{?}{=} 2\|x\|^2 + 2\|y\|^2$$

példa: $\mathbb{R}^2$; $\bar{x} = (1,1)$, $\bar{y} = \left(1,\frac{1}{2}\right)$

$$\left(\|\bar{x}+\bar{y}\|_\infty\right)^2 = \frac{9}{4} = 2,25$$

$$\left(\|\bar{x}-\bar{y}\|_\infty\right)^2 = 1$$

$$2\|\bar{x}\|_\infty^2 = 2$$

$$2\|y\|_\infty^2 = 2$$

$$2+2 \neq 2,25+1$$