I/1. Legyen X egy tetszőleges halmaz és $x, y \in X$;

$$d(x,y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}.$$ Mutassuk meg, hogy (X,d) metrikus tér!

I. $d(x,y) \geq 0$

II. $d(x,y) = 0 \longleftrightarrow x = y$

III. $d(x,y) = d(y,x)$

IV. $d(x,y) \leq d(x,z) + d(z,y)$; $(z \in X)$

I. $d(x,y)$ értéke 1 v. 0 lehet ezért $d(x,y) \geq 0$

II. $d(x,y) = 0$ pontosan akkor, ha $x = y$

III. $d(x,y)$ definíciója miatt teljesül

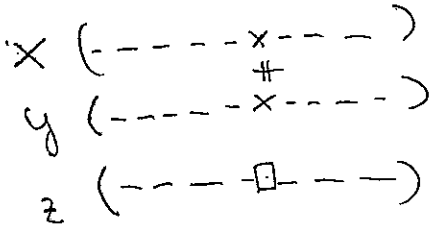
IV. Ha $x = y$ ~~és~~ vagy $y = z$ $\rightarrow$ egyenlőség teljesül

$x \neq y$ ~~és~~ vagy $y \neq z$ $\rightarrow$ $<$ teljesül

vagy $x \neq z$ (legalább az egyik tag $0 <$.)

~~Ha~~

/2. Metrikát definiál-e $\mathbb{R}^n$-en a $d(x,y) = \#\{j; x_j \neq y_j\}$ fv?



I. Mivel $\#\{i; x_i \neq y_i\} \geq 0$, ezért teljesül

II. $\#\{i; x_i \neq y_i\}$ csak akkor 0, ha $\bar{x} = \bar{y}$ ✓

III. $\#\{i; x_i \neq y_i\} = \#\{i; y_i \neq x_i\}$

IV. $d(x,y) \leq d(x,z) + d(z,y)$

Ha $\exists j; x_j \neq y_j \rightarrow (x_j \neq z_j) \vee (y_j \neq z_j)$

teljesül, így a jobb oldalon is legalább 1 koordináta eltérés lesz j. pozícióban

Mutassuk meg, hogy norma:

$\|\cdot\|: V \to \mathbb{R}$, $x, y \in V$, $\lambda \in \mathbb{K}$

1/3.

I. $\|x\| \geq 0$

II. $\|x\| = 0 \Longleftrightarrow x = 0$

III. $\|\lambda x\| = |\lambda| \|x\|$

IV. $\|x+y\| \leq \|x\| + \|y\|$

III. $\sum_{i=1}^{n} |\lambda x_i| = |\lambda| \sum_{i=1}^{n} |x_i|$

IV. $\sum_{i=1}^{n} |x_i + y_i| \leq \sum_{i=1}^{n} |x_i| + \sum_{i=1}^{n} |y_i|$

$\boxed{V = \mathbb{R}^n}$

(a): $\|x\|_1 = \sum_{i=1}^{n} |x_i|$

I. $\sum_{i=1}^{n} |x_i| \geq 0$ II. $\underbrace{\sum_{i=1}^{n} |x_i|}_{\|x\|_1} = 0 \Longleftrightarrow \underbrace{\forall i;\ x_i = 0}_{\bar{x} = \bar{0}}$

(b): CBS: $\sum_{i=1}^{n} |x_i||y_i| \leq \underbrace{\left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}}_{\|x\|_2} \underbrace{\left(\sum_{i=1}^{n} |y_i|^2\right)^{1/2}}_{\|y\|_2}$

$\|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}$

I. Mivel $x_i \in \mathbb{R}\ \forall i$ és $\sum_{i=1}^{n} |x_i|^2 \geq 0$; $\|x\|_2 \geq 0$

II. $\sum_{i=1}^{n} |x_i|^2 = 0 \Longleftrightarrow \forall i\ x_i = 0$

III. $\underbrace{\left(\sum_{i=1}^{n} |\lambda x_i|^2\right)^{1/2}}_{\|\lambda x\|} = \left(|\lambda|^2 \sum_{i=1}^{n} |x_i|^2\right)^{1/2} = |\lambda| \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2} = |\lambda| \|x\|$

IV. $\underbrace{\sum_{i=1}^{n} (|x_i + y_i|)^2}_{\|x+y\|_2^2} \leq \overbrace{\sum_{i=1}^{n} |x_i|^2}^{\|x\|_2^2} + \overbrace{\sum_{i=1}^{n} |y_i|^2}^{\|y\|_2^2} + 2 \sum_{i=1}^{n} |x_i||y_i| \leq$

$\leq \|x\|_2^2 + \|y\|_2^2 + 2\|x\|_2 \|y\|_2 = \left(\|x\|_2 + \|y\|_2\right)^2$

(c)

$\|x\|_\infty = \max_{i=1..n} \{|x_i|\}$

I. $\max_{i=1..n} \{|x_i|\} \geq 0$, mivel $|x_i| \geq 0$

II. $\max_{i=1..n} \{|x_i|\} = 0 \Longleftrightarrow \forall i\ x_i = 0$

III. $\underbrace{\max_{i=1..n} \{|\lambda x_i|\}}_{|\lambda x_i| = |\lambda| \cdot |x_i|} = |\lambda| \max_{i..n} \{|x_i|\}$

IV.

$\left(\cancel{\max_{i=1..n} \{|x_i|\}} = \right)$ $\max_{i=1..n} \{|x_i + y_i|\} \leq \max_{i=1..n} \{|x_i| + |y_i|\}$

$= \max_{i=1..n} \{|x_i|\} + \max_{i=1..n} \{|y_i|\}$

I./4. Legyen $\bar{x} = (1,1,\dots,1) \in \mathbb{R}^n$; számítsuk ki $\bar{x}$ hosszát az I./3. normákon!

(a) $\|\bar{x}\|_1 = \sum_{i=1}^{n} |x_i| = \sum_{i=1}^{n} |1| = n$

(b) $\|\bar{x}\|_2 = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2} = \left(\sum_{i=1}^{n} |1|^2 \right)^{1/2} = \sqrt{n}$

(c) $\|\bar{x}\|_\infty = \max_{i=1\dots n} \{ \overbrace{|x_i|}^{x_i = 1\ \forall i} \} = 1$

$\underline{H}$: $\|\bar{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} = \sqrt[p]{n}$

$\bar{x} = (1,1\dots 1)$ $\qquad \lim_{p\to\infty} \sqrt[p]{n} = 1$

I./6. $V = \mathbb{R}^2$

Rajzoljuk le a: 1. $\|\cdot\|_1$, 2. $\|\cdot\|_2$, 3. $\|\cdot\|_\infty$ szerinti egység gömböt!

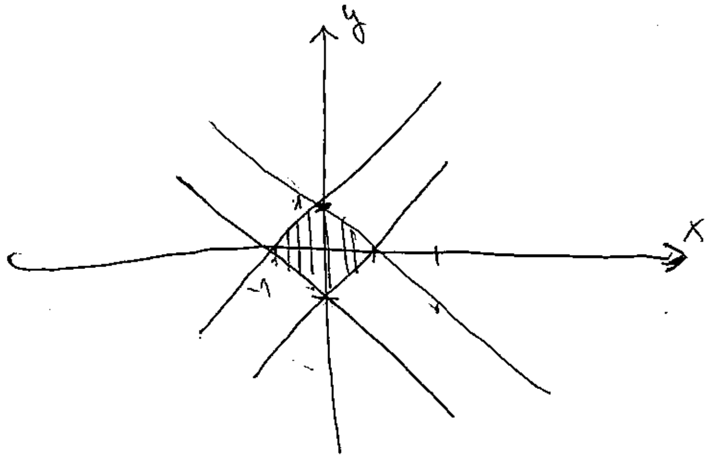
$B_{(1)}(\bar{0}) = \{ \bar{x} \in \mathbb{R}^2 ; \|x\| < 1 \}$

1. $\|\cdot\|_1$; $B_{(1)}(0) = \{ \bar{x} \in \mathbb{R}^2, \underbrace{|x_i + y_i| < 1}\}$

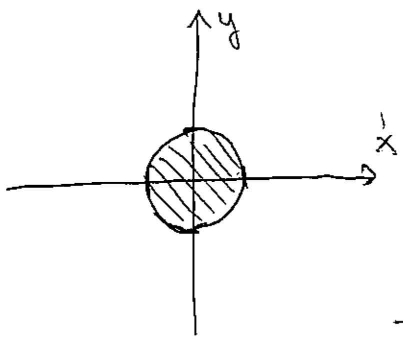
$|x_i + y_i| \leq |x_i| + |y_i| < 1$

$|y_i| < 1 - |x_i|$

+	+
−	−
−	+
+	−



2. $\|\cdot\|_2$; $B_1(0)=\{\bar{x}_i ; \|x\|_2=\sqrt{x_i^2+y_i^2}=1\}$



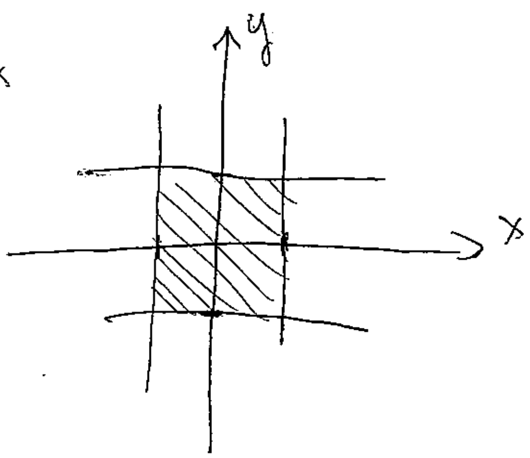
3. $\|\cdot\|_\infty$; $B_1(0)=\{\bar{x} ; \max\limits_{i=1,2}\{|x_i|;|y_1|\}<1\}$

$|x|<1$

$\pm$

$|y|<1$

$\pm$



1./8.

(a): $\forall X ; \exists \|x\| \longrightarrow \exists\ d(x,y), y\in X$?

$\|x\|$ norma $\longrightarrow d(x,0)=\|x-0\|=\|x\|$

$d(x,y)=\|x-y\|$ *

(b) $\forall\ x,y\in X ; (X,d) \longrightarrow \exists$ norma

Pl.: létezik d metrika ami nem származik normából is diszkrét metrika

1./9. $[a,b]\subset\mathbb{R}$; $X=\{f:[a,b]\to\mathbb{R} ; f$ folyt.$\}$

(a): X vektortér? — $f(x)+g(x)\in X$?

— $\lambda f(x)\in X, \lambda\in\mathbb{R}$?

(b) $\|f\|_\infty$.

Analízis I: ~~egy~~ $f,g\in C([a,b])$;

$\longrightarrow f+g\in C([a,b])$

$\longrightarrow \lambda\cdot f\in C([a,b])$

I. $\max\limits_{x\in[a,b]} |f(x)| \geq 0$

(Weierstrass tétel)

II. $\max\limits_{x\in[a,b]} |f(x)|=0 \Longleftrightarrow f(x)=0$

III. $\max\limits_{x\in[a,b]}\{|\lambda f(x)|\}=|\lambda| \max\limits_{x\in[a,b]}\{|f(x)|\}$

I. $\max\limits_{x\in[a,b]}\{|f(x)+g(x)|\} \leq \max\limits_{x\in[a,b]}\{|f(x)|+|g(x)|\} = \|f(x)\|_\infty+\|g(x)\|_\infty$

1./9./(b): $\|f(x)\|_2 = \left(\int_a^b |f^2(x)| \, dx \right)^{1/2}$

I. $\left(\int_a^b |f^2(x)| \, dx \right)^{1/2} \geq 0$

II. $\|f(x)\|_2 = 0 \Longleftrightarrow f(x) = 0$

III. $\|\lambda f(x)\|_2 = \left(\int_a^b |\lambda \cancel{f} f^2(x)| \, dx \right)^{1/2} = |\lambda| \left(\int_a^b |f^2(x)| dx \right)^{1/2}$

IV. $\|f(x) + g(x)\|_2^2 = \int_a^b (|f| + |g|)^2 \, dx = \int_a^b |f|^2 dx + \int_a^b |g|^2 dx$

$+ 2 \underbrace{\int_a^b |\cancel{f(x)}| |g(x)| \, dx}_{\text{CBS}} \leq \underbrace{\int_a^b f^2 dx + \int_a^b g^2 dx + 2 \int f^2 dx \cdot \int g^2 dx}_{(\|f\|_2 + \|g\|_2)^2}$

1./10.

$\forall$ skalárszorzat tér $\to$ normált tér

I. $\|x\| = \underbrace{\langle x, x \rangle}_{N}^{1/2} \geq 0$

II. $\langle x, x \rangle^{1/2} = 0 \Longleftrightarrow x = 0$

III. $\langle \lambda x, \lambda x \rangle^{1/2} = |\lambda| \underbrace{\langle x, x \rangle^{1/2}}_{\|x\|}$

IV. $\|x + y\| =$

$= \langle x + y, x + y \rangle =$

$= \langle x | x + \cancel{y} \rangle + \langle y | x + y \rangle =$

$= \langle x | x \rangle + 2 \langle x | y \rangle + \langle y | y \rangle$

$\wedge$ CBS

$\|x\|^2 + \|y\|^2 + 2 \|x\| \|y\|$

$\|$

$(\|x\| + \|y\|)^2$

11. $(M, d_M), (N, d_N)$

$f: M \to N$ diszkrét metrika

f folytonos x_0-ban: $\forall \varepsilon > 0 \; \exists \delta > 0$:

$\underset{\wedge \, \delta}{d_M(x, x_0)} \to \underbrace{d_N(f(x), f(x_0))}_{= 0} < \varepsilon$

ha $f(x) = f(x_0)$

$\forall x_0, x \in [a, b]$, azaz $f(x) = c$, $c \in \mathbb{R}$