# Computer controlled systems

Lecture 7, March 31, 2017

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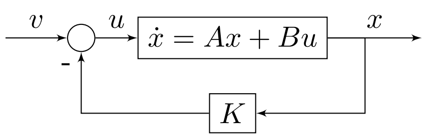
## Pole-placement controller: problem statement

We aim to find a state feedback $u = -Kx$ $(+v)$ such that the closed loop system is stable. Signal $v$ is a possible disturbance, reference, or other extraneous signal.




$$\equiv \quad \dot{x} = (A - BK)x + Bv$$

The state transition matrix of the closed loop system is directly affected by the static state feedback gain $K$. Therefore, $K$ has to be selected such that $A - BK$ has stable poles. This is an algebraic problem.

$$\begin{cases} \dot{x} = Ax + Bu \\ u = -Kx + v \end{cases} \Leftrightarrow \quad \dot{x} = (A - BK)x + Bv \tag{1}$$

### A linear matrix inequality (LMI) solution for MIMO systems

The closed loop is stable if there exists a positive Lyapunov function $V : \mathbb{R}^n \to R$, $V(x) = x^\top P x$, $P = P^\top \succ 0$ such that

$$\tfrac{\partial V}{\partial x}(A - BK)\,x = x^\top \big((A - BK)^\top P + P(A - BK)\big)\,x < 0 \text{ for all } x \neq 0, \tag{2}$$

namely

$$(A^\top - K^\top B^\top)P + P(A - BK) \prec 0, \tag{3}$$

$$A^\top P - K^\top B^\top P + PA - PBK \prec 0. \tag{4}$$

The red color of $K$ and $P$ indicate that these matrices are unknown. Therefore, (4) is a **bilinear** matrix inequality (hard to solve). Let $Q = P^{-1}$, then:

$$Q\big(A^\top P - K^\top B^\top P + PA - PBK\big)Q \prec 0 \tag{5}$$

$$QA^\top - QK^\top B^\top + AQ - BKQ \prec 0. \tag{6}$$

Let $N = KQ$. Then,

$$QA^\top - N^\top B^\top + AQ - BN \prec 0. \tag{7}$$

The matrix inequality in (7) is linear in the (red) unknown matrix variables, therefore, it can be solved with a semidefinite solver (like SeDuMi, Mosek, SDPT3, or PENLAB).

When we do a substitution like $N = KQ$, we should be very careful. The shadowing equation ($N = KQ$) should not be overdetermined with respect to the shadowed variable ($K$). Namely, for any possible pair $(Q, N)$ should exists $K = NQ^{-1} = NP$. In this case, we are fortunate, as $K \in \mathbb{R}^{m \times n}$ and $N \in \mathbb{R}^{m \times n}$, where $(m = \dim(u),\ n = \dim(x))$.

This technique is an optimization based approach. In this formulation we only have to find one possible solution for (7), then, $K = NQ^{-1} = NP$. This computational approach to find $K$ does not involve any sophisticated design objective like select the resulting poles (pole placement controller design), or minimize a cost function (LQR design). However, this technique can be easily extended to a more general class of dynamical systems.

### Pole-placement controller (SISO)

It is given $\dot{x} = Ax + Bu$. A feedback law is searched in the form $u = -Kx$ such that the closed loop $\dot{x} = (A - BK)x$ has a given set of poles.

Design objective:

1. $\det(sI - A) = a(s) = (s - p_1)\cdots(s - p_n) = s^n + a_1 s^{n-1} + \cdots + a_n \quad \to \quad$ unstable poles
2. Design $K$ such that $\det\big(sI - (A - BK)\big) = \alpha(s) = (s - \tilde{p}_1)\cdots(s - \tilde{p}_n) = s^n + \alpha_1 s^{n-1} + \cdots + \alpha_n$.

**Solution 1 (based on the Bass-Gura formula).**

$$\boxed{K = (\underline{\alpha} - \underline{a}) T_l^{-1} \mathcal{C}_n^{-1}}$$

where

$$\underline{\alpha} = \begin{pmatrix} \alpha_1 & \dots & \alpha_n \end{pmatrix}, \; \underline{a} = \begin{pmatrix} a_1 & \dots & a_n \end{pmatrix}, \tag{8}$$

$\mathcal{C}_n = \begin{pmatrix} B & AB & \dots & A^{n-1}B \end{pmatrix}$ is the controllability matrix, finally, $T_l$ is the following Toeplitz matrix:

$$T_l = \begin{pmatrix} 1 & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \\ 0 & 1 & a_1 & \cdots & a_{n-3} & a_{n-2} \\ 0 & 0 & 1 & \cdots & a_{n-4} & a_{n-3} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \cdots & 1 & a_1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

**Solution 2 (the Ackermann formula).**

$$\boxed{K = [0\,0\,\cdots\,0\,1]\,\mathcal{C}_n^{-1}\,\alpha(A)}$$

where $\alpha(s)$ is . Observe that the prescribed characteristic polynomial $\alpha(s)$ of the closed-loop (controlled) system is now evaluated in the state transition matrix, namely:

$$\alpha(A) = A^n + \alpha_1 A^{n-1} + \cdots + \alpha_{n-1} A + \alpha_n I_n. \tag{9}$$

**Example 1.** Design a pole-placement controller for the following CT LTI SISO system:

$$A = \begin{pmatrix} 2 & -2 \\ 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 1 \end{pmatrix}$$

*Solution.*

$$\begin{aligned} a(s) &= s^2 - 3s + 2 \\ a_1 &= -3 \\ a_2 &= 2 \end{aligned}$$

The prescribed characteristic polynomial ($\alpha(s)$):

$$\begin{aligned} \alpha(s) &= s^2 + 3s + 2 \\ \alpha_1 &= 3 \\ \alpha_2 &= 2 \end{aligned}$$

A Toeplitz matrix and the controllability matrix in this case are

$$T_l = \begin{pmatrix} 1 & a_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix} \qquad \mathcal{C} = \begin{pmatrix} 1 & -2 \\ 2 & 2 \end{pmatrix}$$

$$T_l^{-1} = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \qquad \mathcal{C}^{-1} = \frac{1}{6}\begin{pmatrix} 2 & 2 \\ -2 & 1 \end{pmatrix}$$

Than the static state feedback will be the following:

$$K = \begin{pmatrix} 3-(-3) & 2-2 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \frac{1}{6} \begin{pmatrix} 2 & 2 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -4 & 5 \end{pmatrix}$$

**Example 2.** Design a pole-placement controller for the following CT LTI SISO system:

$$A = \begin{pmatrix} 2 & -2 \\ 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 1 \end{pmatrix}$$

*Solution.*

$$\mathcal{C}_2 = \begin{pmatrix} B & AB \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 2 & 2 \end{pmatrix} \to \mathcal{C}_2^{-1} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{6} \end{pmatrix}$$

Legyen $\lambda_1 = -1$ és $\lambda_2 = -2$.

$$\alpha(s) = (s-\lambda_1)(s-\lambda_2) = s^2 + 3s + 2$$

$$\alpha(A) = A^2 + 3A + 2I = \begin{pmatrix} 12 & -12 \\ 0 & 6 \end{pmatrix}$$

$$K = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{6} \end{pmatrix} \begin{pmatrix} 12 & -12 \\ 0 & 6 \end{pmatrix} = \begin{pmatrix} -4 & 5 \end{pmatrix}$$

Check

$$A - BK = \begin{pmatrix} 2 & -2 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} -4 & 5 \end{pmatrix} = \begin{pmatrix} 6 & -7 \\ 8 & -9 \end{pmatrix}$$

$$\det(\lambda I - (A - BK)) = \lambda^2 + 3\lambda + 2$$

Namely, the poles of the obtained closed-loop system are indeed the prescribed values.

**Example 3.** Design a pole-placement controller for the following CT LTI SISO system:

$$A = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 1 \end{pmatrix}$$

*Solution.*

$$\mathcal{C}_2 = \begin{pmatrix} B & AB \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \to \mathcal{C}_2^{-1} = \begin{pmatrix} 1 & -\frac{2}{3} \\ 0 & \frac{1}{3} \end{pmatrix}$$

Let $\lambda_1 = -1$ and $\lambda_2 = -2$.

$$\alpha = (s+\lambda_1)(s+\lambda_2) = s^2 + 3s + 2$$

$$\alpha(A) = A^2 + 3A + 2I = \begin{pmatrix} 9 & -3 \\ 9 & -3 \end{pmatrix}$$

$$K = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{2}{3} \\ 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 9 & -3 \\ 9 & -3 \end{pmatrix} = \begin{pmatrix} 3 & -1 \end{pmatrix}$$

Check:

$$A - BK = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 3 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 3 & -2 \end{pmatrix}$$

$$\det(\lambda I - (A - BK)) = \lambda^2 + 3\lambda + 2$$

Indeed, the poles of the closed loop system are the prescribed values.

**Example 4.** Given the following CT LTI SISO systems

1. $\begin{cases} \dot{x} = \begin{pmatrix} 2 & 0 \\ 9 & -3 \end{pmatrix} x + \begin{pmatrix} 0 \\ 3 \end{pmatrix} u \\ y = \begin{pmatrix} 1 & 1 \end{pmatrix} x \end{cases}$

2. $\begin{cases} \dot{x} = \begin{pmatrix} 2 & 0 \\ 9 & -3 \end{pmatrix} x + \begin{pmatrix} 2 \\ 3 \end{pmatrix} u \\ y = \begin{pmatrix} 1 & 1 \end{pmatrix} x \end{cases}$

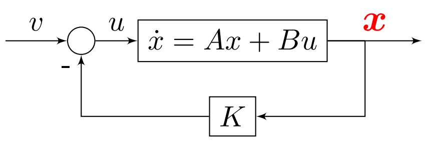
Design a state feedback controller (if it is possible), that stabilizes the system!

**Example 5.** Given the following CT LTI SISO system $H(s) = \frac{2s-4}{s^2+s-6}$.

1. Is the system asymptotically stable?
2. If it is possible, design a controller, that shifts the system's poles to $p_1 = -3$ and $p_2 = -5$! Hint: controllability normal form.

## Linear state observer design

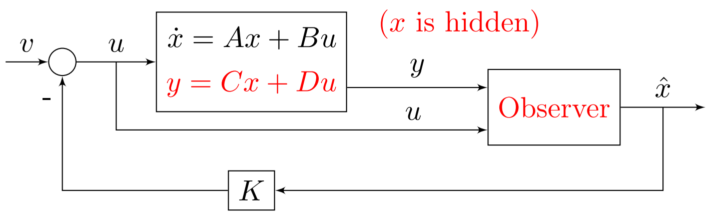
The control approach of the previous sections has a sever problem:




$$\equiv \qquad \dot{x} = (A - BK)x + Bv$$

It is generally very expensive (or completely impossible) to measure the whole state vector.

**Goal**: computation of the values of the non-measured state variables of the system using the observed output.




The dynamical system

$$\dot{\hat{x}} = F\hat{x} + Ly + Hu$$

is called a full order state observer, if the error dynamics $e = x - \hat{x}$ tends to zero, i.e. $\lim_{t\to\infty} e = 0$
In case of an LTI system:

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

$$\dot{e} = \dot{x} - \dot{\hat{x}} = Ax + Bu - F\hat{x} - Ly - Hu + Fx - Fx =$$

$$= Ax + Bu - F\hat{x} - LCx - Hu + Fx - Fx =$$

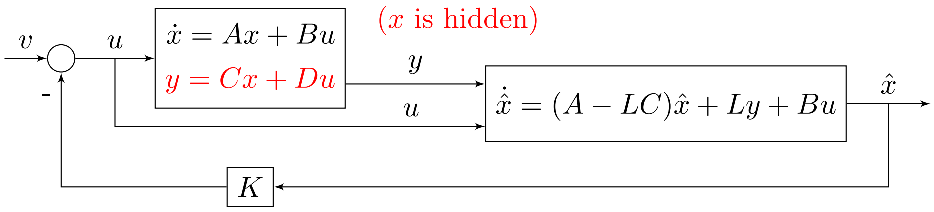
$$= (A - LC - F)x + (B - H)u + F(x - \hat{x}) = (A - LC - F)x + (B - H)u + F(e)$$

Let $F = A - LC$ and $H = B$
Than $\dot{e} = Fe$

We require that the system be asymptotically stable, namely the real part of the roots of the characteristic polynomial $\det(sI - (A - LC))$ be negative.

$$\det(sI - (A - LC)) = \det\big(sI - (A^T - C^T L^T)\big)$$

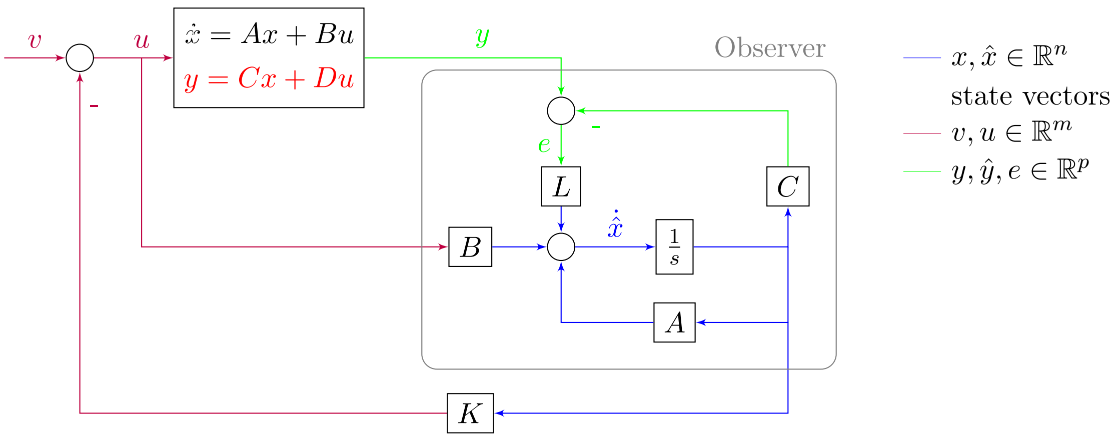
We can observe that the state observer design can be traced back to a pole placement problem of $(A', B')$, where $A' = A^T$, $B' = C^T$, and the result $(K)$ of the pole placement should be interpreted as $L = K^T$.




## The observer dynamics can be considered as follows:

$$\dot{\hat{x}} = \underbrace{A\hat{x} + Bu}_{\text{system dynamics}} + \underbrace{L(y - \hat{y})}_{\text{error term}} \tag{10}$$

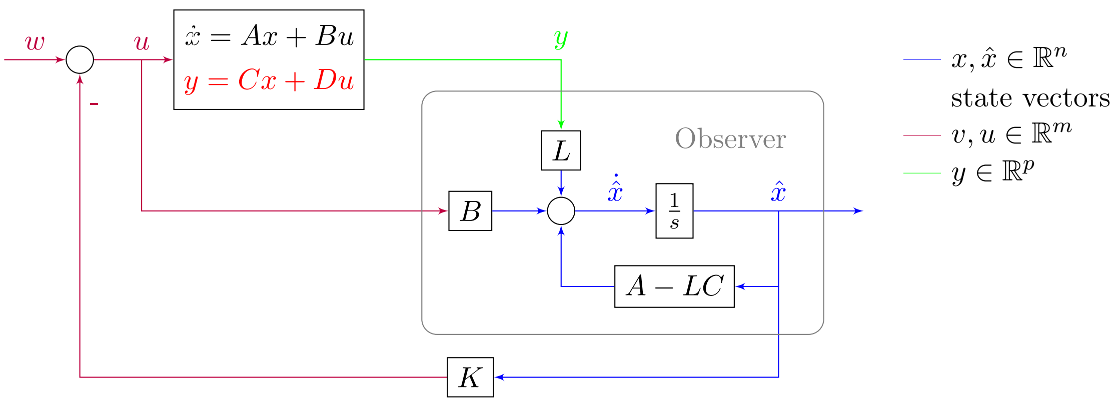
## Corresponding block diagram:




## But also:

$$\dot{\hat{x}} = \underbrace{(A - LC)}_{\text{observer's internal dynamics (should be stable)}} \hat{x} + Bu + Ly \tag{11}$$

## Corresponding block diagram:



**Example 6.** Design a state observer for the following CT LTI SISO system

$$A = \begin{pmatrix} -3 & 1 \\ 2 & -1 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad C = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

*Solution.*
Let the characteristic polynomial of the closed-loop system: $\phi_o(s) = (s+3)(s+3)$
In order to use the Ackermann, formula we should substitute $A' = A^T$ into $\phi_o(s)$:

$$\phi_o(A') = \begin{pmatrix} 2 & 4 \\ 2 & 6 \end{pmatrix}$$

If $B' = C^T$, the obtained controllability matrix for $(A', B')$ (which is actually the transpose of the observability matrix of $(A, C)$) is:

$$\mathcal{C}_2' = \begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix}$$

Its inverse will be:

$$(\mathcal{C}_2')^{-1} = \begin{pmatrix} 1/2 & 1 \\ 1/2 & 0 \end{pmatrix}$$

Finally, we compute the feedback gain $K$:

$$K = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 1/2 & 1 \\ 1/2 & 0 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 2 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 2 \end{pmatrix}$$

From this:

$$L = K^T = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad F = A - LC = \begin{pmatrix} -3 & 0 \\ 2 & -3 \end{pmatrix} \quad H = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

**Example 7.** Design a state observer for the following CT LTI SISO system

$$A = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

**Example 8.** Design a state observer AND a stabilizer state feedback controller for the following CT LTI SISO system.

$$A = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

**Separation principle**: the observer gain $L$ and the feedback gain $K$ can be designed separately.

## Optimal state feedback controller - LQR controller design

We want to minimize the following functional:

$$J(x,u) = \frac{1}{2}\int_0^T x^T Q x + u^T R u \; dt$$

where $Q$ and $R$ are positive definite symmetric matrices. In case of LTI systems this problem can be traced back to a CARE (continuous-time algebraic Riccati equation):

$$\boxed{KA + A^T K - KBR^{-1}B^T K + Q = 0}$$

The system can be stabilized with the $u = -Gx$ state feedback, where

$$\boxed{G = R^{-1}B^T K}$$

**Example 9.** Design an optimal LQR controller for the following system: $\dot{x} = 2x + u$, i.e $A = 2, B = 1$.
*Solution.* We minimize the following functional:

$$J = \frac{1}{2}\int 5x^2 + u^2 dt$$

meaning that in our case $Q = 5$ and $R = 1$. In this case (first order system – only one single state variable) the CARE will have the following form:

$$-K^2 + 4K + 5 = 0$$

The solutions for $K$ are 5 and $-1$. By definition, we should choose the positive one, otherwise, we obtain a positive feedback.

$$G = 1 \cdot 1 \cdot 5 = 5$$

Finally, the computed state feedback: $u = -5x$.