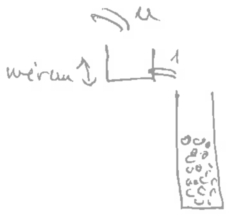
→ BIBO stabilitás (nagyon röviden) $\Big\}$ víz + K. Cola + Mentos pld.

→ LTI stabilitás (nagyon röviden)



$$\underbrace{\begin{aligned}\dot{x} &= \begin{pmatrix}-1 & 0\\ 1 & 1\end{pmatrix}x + \begin{pmatrix}1\\0\end{pmatrix}u\\ y &= (1\ \ 0)\,x\end{aligned}}_{\text{Nem stabil}} \qquad \underbrace{H(s) = \frac{1}{s+1}}_{\text{BIBO stab}}$$

→ Lotka-Volterra:

$$\dot{x}_1 = +7x_1 - 4x_1^2 - 3x_1x_2 \quad \text{(nyulak/zsákmány)}$$

$$\dot{x}_2 = -3x_2 + 2x_1x_2 + x_2^2 \quad \text{(farkas/ragadozó)}$$

↳ $x^*_{(0)} = \begin{pmatrix}0\\0\end{pmatrix}$ → instab

↳ $x^*_{(1)} = \begin{pmatrix}1\\1\end{pmatrix}$ → stabil

→ Inga (sima) + Ljapunov függvény

$$\dot{\theta} = \omega$$

$$\dot{\omega} = -\frac{g}{l}\sin\theta - b\omega$$

$$V = mgl(1-\cos\theta) + \frac{ml^2\omega^2}{2}$$

(Szimuláció + videó)

→ (Inga Hamilton rendszer) ← erre talán nem lesz idő

$$\dot{q} = \frac{\partial H}{\partial p}$$

$$\dot{p} = -\frac{\partial H}{\partial q} - bp \quad \text{ahol} \quad p = \frac{\partial L}{\partial \dot{q}}$$

→ Van der Pol oscillátor (paraméteres Ljp.) +optim.

$$\begin{cases}\dot{x}_1 = -x_2\\ \dot{x}_2 = +x_1 + (1-x_1^2)x_2\end{cases}$$

$$V = x^T\begin{pmatrix}a & \\ & b\end{pmatrix}x$$

Lotka – Volterra: $\dot{x} = \mathrm{diag}(x)(Ax+b) \Rightarrow x^* = -A^{-1}b$

$b = -Ax^*$

$A = \begin{pmatrix} -2 & -3 \\ 1.4 & 1 \end{pmatrix} \Rightarrow b = -\begin{pmatrix} -2 & -3 \\ 1.4 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = -\begin{pmatrix} -5 \\ 2.4 \end{pmatrix}$

$x^* = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\begin{cases} \dot{x}_1 = x_1(-2x_1 - 3x_2 + 5) & \text{nyulak} \\ \dot{x}_2 = x_2(1.4x_1 + x_2 - 2.4) & \text{ragadozók} \end{cases}$$

Egyensúlyi pontok: $x^*_{(0)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $x^*_{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

átírva:

$$\begin{cases} \dot{x}_1 = -2x_1^2 - 3x_1x_2 + 5x_1 \\ \dot{x}_2 = 1.4x_1x_2 + x_2^2 - 2.4x_2 \end{cases} \qquad f(x) = \begin{pmatrix} -2x_1^2 - 3x_1x_2 + 5x_1 \\ 1.4x_1x_2 + x_2^2 - 2.4x_2 \end{pmatrix}$$

$$\frac{\partial f}{\partial x} = \begin{pmatrix} -4x_1 - 3x_2 + 5 & -3x_1 \\ 1.4x_2 & 1.4x_1 + 2x_2 - 2.4 \end{pmatrix}$$

$$\frac{\partial f}{\partial x}(0,0) = \begin{pmatrix} 5 & \\ & -2.4 \end{pmatrix}$$ tehát az $x^*_{(0)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ nem stabil egy. pont

$$\frac{\partial f}{\partial x}(1,1) = \begin{pmatrix} -2 & -3 \\ 1.4 & 1 \end{pmatrix}$$

$$\det\begin{pmatrix} s+2 & 3 \\ -1.4 & s-1 \end{pmatrix} = s^2 + s - 2 + 4.2$$

$$= s^2 + s + 2.2$$

$s_{1,2} = -\frac{1}{2} \pm 1.396\,i$

$\Downarrow$

stabil

$\Delta = 1 - 8.8 = -7.8$

Megjegyzés: $A = \begin{pmatrix} -4 & -3 \\ 2 & 1 \end{pmatrix}$-re $s_{1,2} \in \{-1, -2\}$.

2020b

Nemlineáris rendszer lokális stabilitása.

$$\boxed{\ddot{\theta} + b\dot{\theta} + \frac{g}{l}\sin\theta = 0}$$

Pendulum ← Ljapunov fv.

$$\begin{cases} x_1 = \theta \\ x_2 = \dot{\theta} \end{cases} \Rightarrow \begin{cases} \dot{x}_1 = \dot{\theta} = x_2 \\ \dot{x}_2 = \ddot{\theta} = -\frac{g}{l}\sin\theta - b\dot{\theta} = -\frac{g}{l}\sin x_1 - bx_2 \end{cases}$$

$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ akkor $\dot{x} = f(x)$, ahol $f(x) = \begin{pmatrix} x_2 \\ -\frac{g}{l}\sin x_1 - bx_2 \end{pmatrix}$

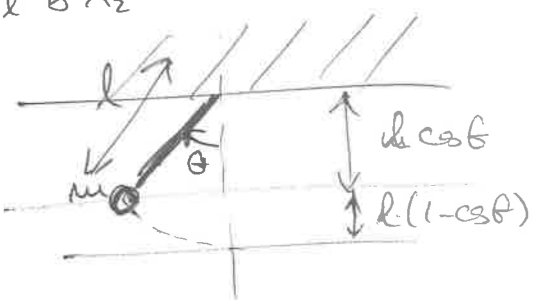
Legyen $V(x_1, x_2) = \frac{ml^2 x_2^2}{2} + mgl(1-\cos x_1)$

$$\dot{V} = \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 = \left(\frac{\partial V}{\partial x} f(x)\right)$$

$$= (mgl\sin x_1)\cdot x_2 + (ml^2 x_2)\cdot\left(-\frac{g}{l}\sin x_1 - bx_2\right)$$

$$= mgl\, x_2 \sin x_1 - mgl\, x_2 \sin x_1 - ml^2 b x_2^2$$

$$= -ml^2 b x_2^2$$



$$\boxed{\dot{V} = -ml^2 b x_2^2} < 0$$

damping coefficient!

The Hamiltonian mechanics can note why the total energy is the best Ljapunov func?

$$\begin{cases} \dot{q} = \frac{\partial \mathcal{H}}{\partial p} \\ \dot{p} = -\frac{\partial \mathcal{H}}{\partial q} - bp \end{cases}$$

(Hamiltonian model of the pendulum)

Let $V = \mathcal{H} = \frac{p^2}{2ml^2} + mgl(1-\cos q)$

$$\dot{V} = \frac{\partial V}{\partial p}\dot{p} + \frac{\partial V}{\partial q}\dot{q} = \frac{\partial \mathcal{H}}{\partial p}\dot{p} + \frac{\partial \mathcal{H}}{\partial q}\dot{q} =$$

$$= \underbrace{\frac{\partial \mathcal{H}}{\partial p}\left(-\frac{\partial \mathcal{H}}{\partial q}\right.}\left. - bp\right) + \underbrace{\frac{\partial \mathcal{H}}{\partial q}\frac{\partial \mathcal{H}}{\partial p}} = -(ml^2)^{-1} b p^2 < 0$$

miss each other

$$\mathcal{L} = T - V = \frac{ml^2\omega^2}{2} - mgl(1-\cos\theta) = \frac{ml^2\dot{q}^2}{2} - mgl(1-\cos q)$$

legyen $q = \theta$ (jel: ált. coordináta)

legyen $\dot{q} = \dot{\theta} = \omega$ (ált. sebesség)

legyen $p = \frac{\partial \mathcal{L}}{\partial \dot{q}} = ml^2\dot{q}$ (általánosított momentum)

legyen $\mathcal{H} = \dot{q}\cdot p - \mathcal{L} = \dot{q}p - \left(\frac{ml^2\dot{q}\cdot\dot{q}}{2} - mgl(1-\cos q)\right)$

$$= \frac{ml^2\dot{q}\cdot\dot{q}}{2} + mgl(1-\cos q)$$

$$= \frac{p^2}{2ml^2} + mgl(1-\cos q)$$

sum of kinetic and potential energy in the terms of the generalized coordinate (q)

— " — momentum (p)

First order DE of the motion:

$$\begin{cases} \dot{q} = +\frac{\partial \mathcal{H}}{\partial p} \\ \dot{p} = -\frac{\partial \mathcal{H}}{\partial q} \end{cases} \Longleftrightarrow$$

Euler–Lagrange equations:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = 0$$

$$\Downarrow$$

$$\frac{d}{dt}\left(ml^2\dot{q}\right) + mgl\sin q = 0$$

$$\Downarrow$$

$$ml^2\ddot{q} + mgl\sin q = 0$$

$$\Downarrow$$

$$\boxed{\ddot{q} + \frac{g}{l}\sin q = 0}$$

$$\Downarrow$$

$$\boxed{\begin{cases} \dot{q} = \frac{p}{ml^2} \\ \dot{p} = -mgl\sin q \end{cases}}$$

$$\Downarrow$$

$$\ddot{q} = \frac{\dot{p}}{ml^2} = -\frac{g}{l}\sin q$$

$$\Downarrow$$

$$\boxed{\ddot{q} + \frac{g}{l}\sin q = 0}$$

20206

Hamiltonian ~~model~~ model of pendulum

Van der Pol lokális Ljapunov fv. (2020b)

$$\begin{cases} \dot{x}_1 = -x_2 \\ \dot{x}_2 = +x_1 - (1-x_1^2)x_2 \end{cases} \Rightarrow \begin{cases} \dot{x}_1 = -x_2 \\ \dot{x}_2 = x_1 - x_2 + x_1^2 x_2 \end{cases}$$

Legyen $V(x) = x^T \begin{pmatrix} a & \\ & b \end{pmatrix} x$ paraméteres Ljapunov fv. jelölt

$a, b = ?$ ú.h. (legalább lokálisan) $V(x)$ L.f. legyen

$$\frac{\partial V}{\partial x} f(x) = x^T \begin{pmatrix} a & \\ & b \end{pmatrix} \begin{pmatrix} -x_2 \\ x_1 - x_2 + x_1^2 x_2 \end{pmatrix} =$$

$$= (x_1 a \quad x_2 b) \cdot \begin{pmatrix} \cdots \\ \cdots \end{pmatrix} = \underbrace{-x_1 x_2 a + x_1 x_2 b}_{\text{ha } a=b \text{ akkor ezek kiejthetők}} - x_2^2 b + x_1^2 x_2^2 b$$

Legyen $a = b = 1$, azaz $V(x) = x^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x = x^T x = x_1^2 + x_2^2$

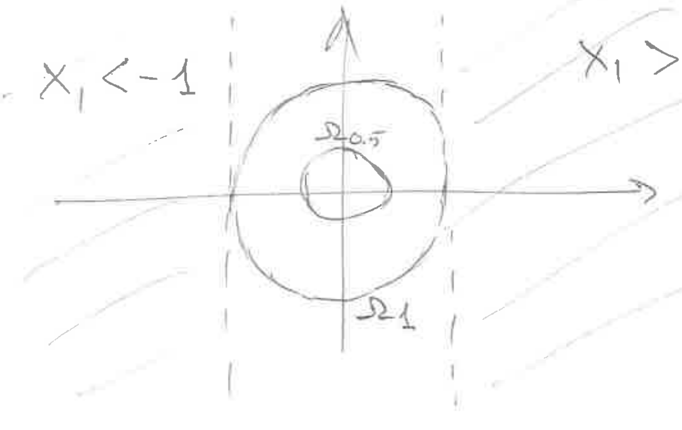
akkor:

$$\frac{\partial V}{\partial x} f(x) = -x_2^2 + x_1^2 x_2^2 = -x_2^2 \underbrace{(1 - x_1^2)}_{\text{tfh } |x_1| < 1} < 0 \text{ ha}$$

Invariáns tartomány (stabil régió):

$$\Omega_c = \{ x \in \mathbb{R}^2 \mid V(x) \le c \} \subseteq \{ x \in \mathbb{R} \mid |x_1| < 1 \}$$

↳ Megengedett régió (ahol $\frac{\partial V}{\partial x} f(x) < 0$)



$\boxed{\Omega_{1-\varepsilon} \text{ teljes egészében benne van}}$

2020b.

Van der Pol Lokális Ljapunov fv.