

Computer controlled systems

Lecture 4

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1 State space transformation

As we shall already know, the state space model is not unique. For the given example, define a new SSM using a state space transformation.

$$A = \begin{pmatrix} -6 & -4 \\ 2 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \quad C = (0 \quad 1)$$

Let the linear transformation of the state vector be the following:

$$\begin{aligned} \bar{x}_1 &= x_1 + x_2 \\ \bar{x}_2 &= 3x_1 - 2x_2 \end{aligned}$$

In matrix form:

$$\begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 3 & -2 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$\bar{x} = Tx$, $x = T^{-1}\bar{x}$ → state space equation can be written for the new state vector \bar{x} as well

$$\dot{x} = Ax + Bu \quad \rightarrow \quad T^{-1}\dot{\bar{x}} = AT^{-1}\bar{x} + Bu$$

$$\dot{\bar{x}} = TAT^{-1}\bar{x} + TBu \quad \rightarrow \quad \bar{A} = TAT^{-1} \quad \bar{B} = TB$$

$$y = Cx = CT^{-1}\bar{x} \quad \rightarrow \quad \bar{C} = CT^{-1}$$

Returning to the example:

$$T = \begin{pmatrix} 1 & 1 \\ 3 & -2 \end{pmatrix} \quad T^{-1} = -\frac{1}{5} \begin{pmatrix} -2 & -1 \\ -3 & 1 \end{pmatrix}$$

$$\bar{A} = TAT^{-1} = \begin{pmatrix} -4 & 0 \\ -16 & -2 \end{pmatrix} \quad \bar{B} = TB = \begin{pmatrix} 4 \\ 12 \end{pmatrix} \quad \bar{C} = CT^{-1} = \left(\frac{3}{5} \quad -\frac{1}{5}\right)$$

If the original and the transformed SSM are (A, B, C) and $(\bar{A}, \bar{B}, \bar{C})$, respectively, determine the transformation matrix T , which connects them.

$$A = \begin{pmatrix} 3 & 2 \\ -4 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad C = (1 \quad 0) \tag{1}$$

$$\bar{A} = \begin{pmatrix} 1.8 & 1.6 \\ -4.4 & 2.2 \end{pmatrix} \quad \bar{B} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad \bar{C} = (0.4 \quad -0.2) \tag{2}$$

Solution. $\bar{B} = TB$, $\bar{A}\bar{B} = TAB$ → $T \cdot [B|AB] = [\bar{B}|\bar{A}\bar{B}]$ → $T = \bar{C}_2 \cdot C_2^{-1}$, where $C_2 = [B|AB]$ and $\bar{C}_2 = [\bar{B}|\bar{A}\bar{B}]$ are the controllability matrices of (1) and (2), respectively.

Remark. B and AB are (2×1) matrices.

$$C_2 = \begin{pmatrix} 1 & 5 \\ 1 & -3 \end{pmatrix}, \quad C_2^{-1} = \frac{1}{-8} \begin{pmatrix} -3 & -5 \\ -1 & 1 \end{pmatrix}, \quad \bar{C}_2 = \begin{pmatrix} 3 & 7 \\ 1 & -11 \end{pmatrix}$$

$$T = \frac{1}{-8} \cdot \begin{pmatrix} 3 & 7 \\ 1 & -11 \end{pmatrix} \cdot \begin{pmatrix} -3 & -5 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \quad T^{-1} = \frac{-1}{5} \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$$

Just as in the previous example, determine the transformation matrix T .

$$A = \begin{pmatrix} 2 & 0 \\ -1 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad C = (1 \ 0) \quad (3)$$

$$\bar{A} = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} \quad \bar{B} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \bar{C} = \left(\frac{-1}{2} \quad \frac{1}{2}\right) \quad (4)$$

Solution. $T = \begin{pmatrix} -1 & 2 \\ 1 & 2 \end{pmatrix} \quad T^{-1} = \frac{-1}{4} \cdot \begin{pmatrix} 2 & -2 \\ -1 & -1 \end{pmatrix}$

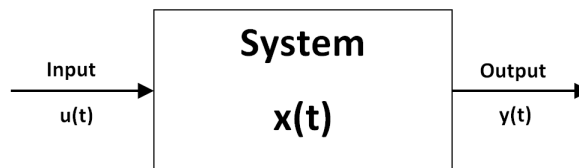
Remark. In case of SISO model, this method can be applied for an even higher dimensional state-space model, but then the controllability matrix will involve further rows. If the state vector is n -dimensional ($A \in \mathbb{R}^{n \times n}$), then $\mathcal{C}_n = [B|AB|A^2B|\dots|A^{n-1}B]$. To conclude, if the SSM is controllable:

$$T = \bar{\mathcal{C}}_n \cdot \mathcal{C}_n^{-1} \quad (5)$$

Megjegyzés: SISO modell esetén a fenti módszer több állapotváltozó esetén is alkalmazható, de ekkor több oszlopra van szükség. Ha $A \in \mathbb{R}^{n \times n}$, akkor a $[B|AB|A^2B|\dots|A^{n-1}B]$ alakú mátrixokkal lehet számolni.

2 Controllability, observability

In general Given the following CT-LTI system: The question arise: In the full knowledge of $y(t)$ and



$u(t)$ can we say something about the unknown state vector $x(t)$? In the other words is $x(t)$ **observable**?

The second question would be the following: is there an input function $u(t)$, with which we can lead the system from the initial state x_0 to state x_1 in a finite time. If we can do so (for every possible initial and final states), we say that the system is **controllable**.

2.1 Observability

Theorem 1.

Sufficient and necessary condition for observability

A state space model described by matrices (A, B, C) is observable if and only if (iff) its observability matrix \mathcal{O}_n is full-rank:

$$\mathcal{O}_n = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}, \quad \text{rank}(\mathcal{O}_n) = n$$

Remark. In SISO case \mathcal{O}_n is a square matrix, which is full-rank iff its determinant is nonzero.

Example 1. Is the system (A, B, C) observable?

$$A = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad C = (0 \quad 1)$$

The observability matrix is the following

$$CA = (2 \quad 1) \rightarrow \mathcal{O}_2 = \begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}, \quad \det(\mathcal{O}_2) = -2 \neq 0 \Rightarrow \mathcal{O}_2 \text{ is full-rank}$$

Hence, $x(t)$ is observable, namely, using $y(t)$ and its time derivative $\dot{y}(t)$, we can compute the actual value of $x(t)$

$$\begin{cases} y(t) = Cx(t) \\ \dot{y}(t) = CAx(t) + CBu(t) \end{cases} \Rightarrow x(t) = \mathcal{O}_2^{-1} \begin{pmatrix} y(t) \\ \dot{y}(t) - CBu(t) \end{pmatrix} \quad (6)$$

Example 2. Unobservable subspace (mathematical background presented in B.1)

Given the state space model:

$$A = \begin{pmatrix} 1 & 2 \\ -2 & -3 \end{pmatrix}, \quad B : \text{arbitrary}, \quad C = (1 \quad 1), \quad \mathcal{O}_n = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \quad (7)$$

A basis for the kernel of \mathcal{O}_n is $v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. This means that

- if there is a zero input and $x(0) = \lambda v_1 \in \text{Ker}(\mathcal{O}_2)$, then $x(t) \in \text{Ker}(\mathcal{O}_2)$ (Proposition 9) and $y(t) = 0$ for every $t > 0$.
- for a given input $u(t)$ and with an initial condition $x(0) = x_0 + \lambda v_1 \in x_0 + \text{Ker}(\mathcal{O}_2)$ (where $\lambda \in \mathbb{R}$ is arbitrary) the system will produce *the same output* $y(t)$.

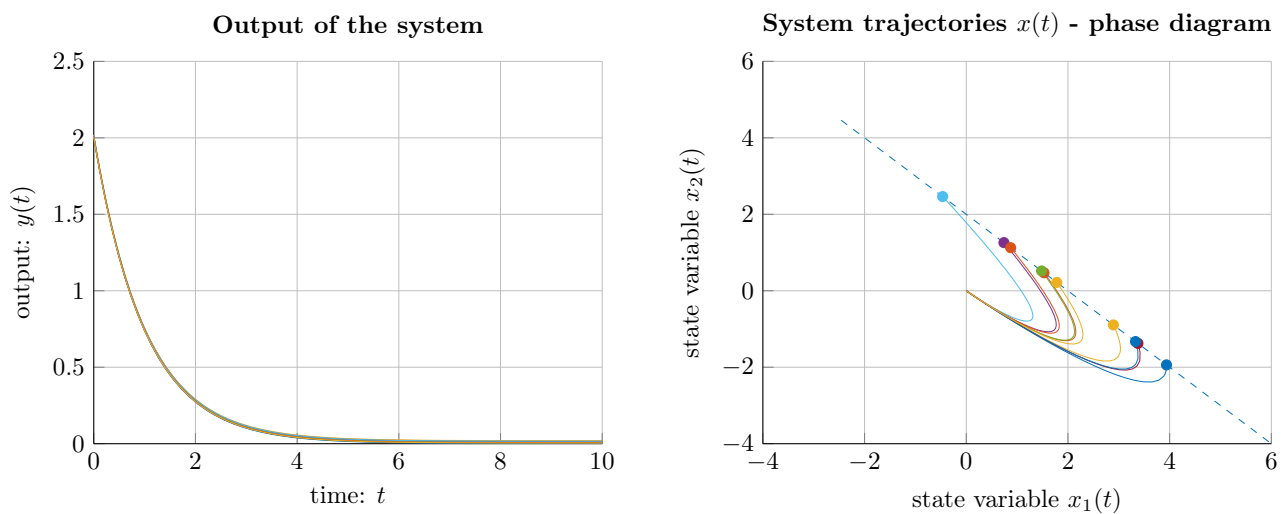


Figure 1. Simulation of system (7) from different initial conditions $x(0) \in x_0 + \text{Ker}(\mathcal{O}_2)$ (denoted by dots) with zero input. As one can observe, the state trajectories are different, however this difference does not appear in the output of the system. In this example $u \equiv 0$ and $x_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. The blue dashed line in the right figure illustrates the actual unobservability subspace of the system corresponding to x_0 .

2.2 Controllability

Given a strictly proper state space model (A, B, C) with $x(t_0)$ initial and $x(t_1) \neq x(t_0)$ final condition. The question arises, is there any input function $u(t)$, which leads the system from $x(t_0)$ to $x(t_1)$ in a finite time.

Theorem 2.

Controllability

A state space model described by matrices (A, B, C) is controllable iff its controllability matrix \mathcal{C}_n is full-rank:

$$\mathcal{C}_n = (B \ AB \ \dots \ A^{n-1}B), \quad \text{rank}(\mathcal{C}_n) = n$$

Remark. In SISO case \mathcal{C}_n is a square matrix, which is full-rank iff its determinant is nonzero.

Example 3.

$$A = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad C = (0 \ 1), \quad \mathcal{C}_2 = (B \ AB) = \begin{pmatrix} 1 & 5 \\ 1 & 3 \end{pmatrix}$$

This system is controllable, since the determinant of \mathcal{C}_2 is nonzero. In this case the controllability subspace is the whole \mathbb{R}^2 itself. If we start the system from zero initial condition, we can lead the system (with an appropriate input) to any other states of the controllability subspace in a finite time.

Example 4. Controllable subspace (mathematical background presented in B.2)

Given the following state space system and its rank-deficient controllability matrix:

$$A = \begin{pmatrix} -1 & 2 & -2 \\ -\frac{2}{3} & -6 & \frac{20}{3} \\ -\frac{1}{2} & -1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 8 \\ 0 \end{pmatrix}, \quad \text{eigenvalues of } A: \begin{pmatrix} -2 \\ -2 \\ -4 \end{pmatrix}, \quad \mathcal{C}_3 = \begin{pmatrix} 0 & 16 & -96 \\ 8 & -48 & 224 \\ 0 & -8 & 48 \end{pmatrix} \quad (8)$$

The basis vectors of $\text{Im}(\mathcal{C}_3)$ are: $v_1 = \begin{pmatrix} 0.3832 \\ -0.9036 \\ -0.1916 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0.8082 \\ 0.4285 \\ -0.4041 \end{pmatrix}$. They span a 2-dimensional subspace in \mathbb{R}^3 , illustrated by the green plane in the Figure 2. If we start the system from an initial condition which is an element of this subspace $x(0) \in \text{Im}(\mathcal{C}_3)$, the system trajectory will never leave this subspace. If the initial condition is outside of $\text{Im}(\mathcal{C}_3)$ and A is stable, the system trajectory will tend to this subspace.

System trajectories $x(t)$ - phase diagram

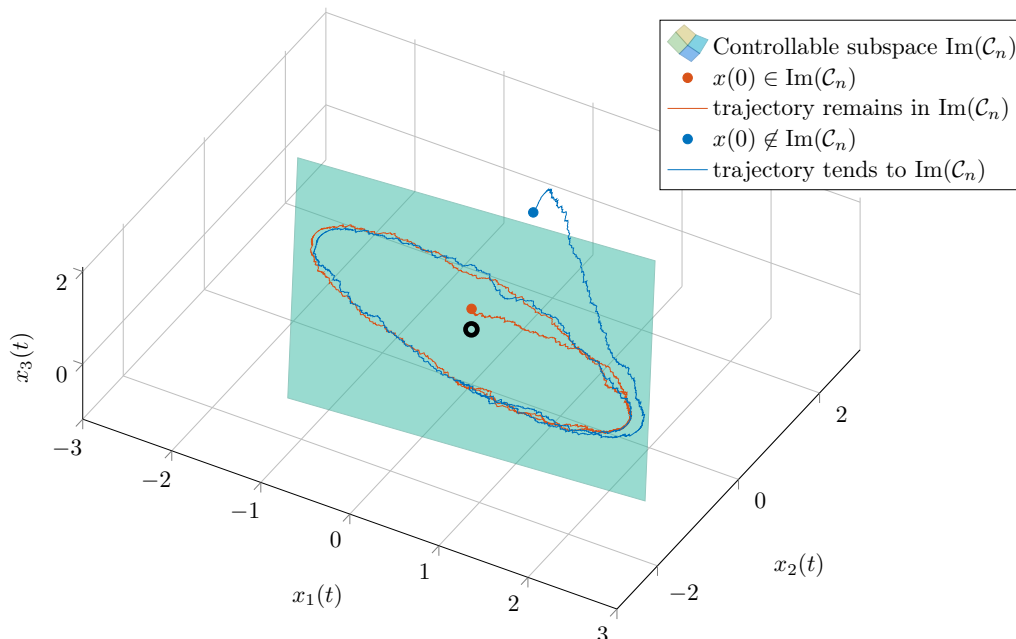


Figure 2. Simulation of system (7) from different initial conditions

Example 5.

Compute the controllable subspace of $\dot{x} = Ax + Bu$, where

$$A = \begin{pmatrix} 1 & 2 & -2 \\ -0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \quad (9)$$

To check your solutions, we give:

$$A^2 = \begin{pmatrix} -1 & 4 & -4 \\ 0 & 1 & 0 \\ 2 & 2 & -1 \end{pmatrix}, \mathcal{O}_3 = \begin{pmatrix} -1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & -2 \end{pmatrix}. \quad (10)$$

2.3 Controllability and observability in case of a diagonal SSM

$$A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \quad B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \quad AB = \begin{pmatrix} a_1 b_1 \\ a_2 b_2 \end{pmatrix} \quad C = (c_1 \quad c_2) \quad CA = (c_1 a_1 \quad c_2 a_2)$$

$$\mathcal{C}_2 = \begin{pmatrix} b_1 & a_1 b_1 \\ b_2 & a_2 b_2 \end{pmatrix} \quad \mathcal{O}_2 = \begin{pmatrix} c_1 & c_2 \\ c_1 a_1 & c_2 a_2 \end{pmatrix}$$

SISO rendszer diagonális A mátrix esetén

irányítható \iff a főátlóbeli elemek páronként különbözőek, és $\forall i \ b_i \neq 0$

megfigyelhető \iff a főátlóbeli elemek páronként különbözőek, és $\forall j \ c_j \neq 0$

Theorem 3. The rank of \mathcal{O}_n and \mathcal{C}_n is invariant to the state space transformations.

Proof.

$$\bar{A} = TAT^{-1} \quad \bar{B} = TB \quad \bar{C} = CT^{-1}$$

$$\bar{\mathcal{C}}_n = (TB \quad TAT^{-1}TB) = T(B \quad AB) = T\mathcal{C}_n$$

$$\bar{\mathcal{O}}_n = \begin{pmatrix} CT^{-1} \\ CT^{-1}TAT^{-1} \end{pmatrix} = \begin{pmatrix} C \\ CA \end{pmatrix} T^{-1} = \mathcal{O}_n T^{-1}$$

□

2.4 Markov parameters

$$CA^i B$$

Markov parameters are invariant to the state space transformations.

$$\bar{C}B = CT^{-1}TB = CB$$

$$\bar{C}\bar{A}\bar{B} = CT^{-1}TAT^{-1}TB = CAB$$

3 Joint controllability and observability

- Egy $H(s) = \frac{b(s)}{a(s)}$ (SISO) átviteli függvény n -edrendű realizációjának nevezzük az (A, B, C, D) állapotér-modellt, ha $H(s) = C(sI - A)^{-1}B + D$, ahol $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times 1}$, $C \in \mathbb{R}^{1 \times n}$, $D \in \mathbb{R}$ (nem egyértelmű!)
- Egy $H(s)$ átviteli függvény n -edrendű realizációját minimálisnak nevezzük, ha nem létezik nála kisebb rendű realizáció.

- Egy n -dimenziós (A, B, C, D) állapotér-modellt együttesen irányíthatónak és megfigyelhetnek nevezünk, ha teljesülnek rá az irányíthatóság és a megfigyelhetőség feltételei (azaz \mathcal{O}_n és \mathcal{C}_n teljes rangú).
- Egy ÁTM minimális \iff egyszerre irányítható és megfigyelhető.

Example 6. Is the state space representation minimal?

$$A = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad C = (0 \quad 1)$$

Transfer function: $H(s) = \frac{s}{s^2 - 3s - 4}$. This SSM is minimal, since $H(s)$ is irreducible and the degree of the denominator is equal to the order of the state space realization ($n = 2$).

Example 7. Is the state space representation minimal?

$$A = \begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad C = (1 \quad 1)$$

$$H(s) = C(sI - A)^{-1}B = \frac{s+1}{s^2 + 4s + 3} = \frac{s+1}{(s+1)(s+3)}$$

This SSM is not minimal, meaning the one of two properties is broken: the SSM is controllable but its is no observable.

Example 8. Is the state space representation minimal?

$$A = \begin{pmatrix} -6 & -4 \\ 2 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \quad C = (0 \quad 1)$$

Controllability matrix:

$$\mathcal{C}_2 = (B \quad AB) = \begin{pmatrix} 4 & -24 \\ 0 & 8 \end{pmatrix}$$

The determinant of matrix \mathcal{C}_2 is nonzero, therefore, it is controllable.

Observability matrix:

$$\mathcal{O}_2 = \begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$$

The determinant of matrix \mathcal{O}_2 is nonzero, therefore, it is observable. Consequently, the SSM is minimal.

Example 9. (MIMO case) Is the state space representation minimal?

$$A = \begin{pmatrix} 1 & 4 \\ -2 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 4 & 1 \\ 2 & 3 & 0 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 2 \\ 0 & 7 \end{pmatrix}$$

$$AB = \begin{pmatrix} 9 & 16 & 1 \\ 2 & -2 & -2 \end{pmatrix} \quad CA = \begin{pmatrix} -3 & 8 \\ -14 & 14 \end{pmatrix}$$

$$\mathcal{O}_2 = \begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 7 \\ -3 & 8 \\ -14 & 14 \end{pmatrix} \quad \mathcal{C}_2 = (B \quad AB) = \begin{pmatrix} 1 & 4 & 1 & 9 & 16 & 1 \\ 2 & 3 & 0 & 2 & -2 & -2 \end{pmatrix}$$

Matrix \mathcal{O}_2 is full-column-rank, and \mathcal{C}_2 is full row rank, meaning that the system is jointly controllable and observable and (A, B, C) is minimal.

Example 10. Is the SSM minimal? If not give a minimal representation.

$$A = \begin{pmatrix} -3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} \quad C = (3 \quad 0 \quad 4)$$

$$H(s) = \sum_{i=1}^n \frac{c_i b_i}{s - \lambda_i} = \frac{3 \cdot 1}{s+3} + \frac{0 \cdot 2}{s-4} + \frac{6 \cdot 4}{s-6} = \frac{3(s-6) + 24(s+3)}{(s+3)(s-6)}$$

$$H(s) = \frac{27s + 54}{s^2 - 3s - 18}$$

The SSM is not minimal, because the transfer function can be reduced.

$$A^2 = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 36 \end{pmatrix} \quad \mathcal{O}_n = \begin{pmatrix} 3 & 0 & 4 \\ -9 & 0 & 24 \\ 27 & 0 & 144 \end{pmatrix} \quad C_n = \begin{pmatrix} 1 & -3 & 9 \\ 2 & 8 & 32 \\ 6 & 36 & 216 \end{pmatrix}$$

A minimal SSM can be given by skipping the single degenerated state variable:

$$A = \begin{pmatrix} -3 & 0 \\ 0 & 6 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 6 \end{pmatrix} \quad C = (3 \quad 4)$$

A minimal realization can also be given using the controller form:

$$A = \begin{pmatrix} 3 & 18 \\ 1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad C = (27 \quad 54)$$

Example 11. It is given a SSM in the controller form. Is the SSM jointly controllable and observable?

$$A = \begin{pmatrix} 0 & 7 & -6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad C = (0 \quad 3 \quad 9)$$

Transfer function:

$$H(s) = \frac{3s + 9}{s^3 - 7s + 6}$$

The realization is most be controllable, since it is given in controller form:

$$A^2 = \begin{pmatrix} 7 & -6 & 0 \\ 0 & 7 & -6 \\ 1 & 0 & 0 \end{pmatrix} \quad \mathcal{O}_n = \begin{pmatrix} 0 & 3 & 9 \\ 3 & 9 & 0 \\ 9 & 21 & -18 \end{pmatrix} \quad C_n = \begin{pmatrix} 1 & 0 & 7 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{array}{l} \text{rank}(C_n) = 3 \\ \text{rank}(\mathcal{O}_n) = 2 \end{array}$$

However the SSM is not observable, because it is not minimal: $H(s)$ is reducible by $s + 3$. Using the controller form (on the irreducible form of $H(s)$), we can obtain a jointly controllable and observable realization. Tehát nem együttesen megfigyelhető és irányítható a rendszer. The a unobservable subspace

$$\text{Ker}(\mathcal{O}_n) = \left\{ \alpha \begin{pmatrix} 9 \\ -3 \\ 1 \end{pmatrix} \mid \alpha \in \mathbb{R} \right\}$$

Felhasználás: Állapotmegfigyelők tervezése

Bizonyos mennyiségeket (pl. szögsebesség) nem tudunk mérni, csak becsülni. Ld.: 3. ábra

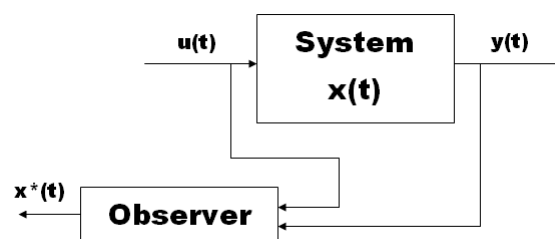


Figure 3. State observer design

References

- [1] Alexey Grigorev. [The Fundamental Theorem of Linear Algebra](#). Technische Universität Berlin.

- [2] Lantos Béla. *Irányítási rendszerek elmélete és tervezése I.* Akadémiai Kiadó Budapest, 2001.
- [3] A. D. Lewis. *A Mathematical Approach to Classical Control.* 2003.

A Supplementary material in linear algebra (not needed for the exam)

Theorem 4.

The fundamental theorem of linear algebra

Let $\mathcal{A} : \mathbb{R}^n \rightarrow \mathbb{R}^m$, $\mathcal{A}(x) = Ax$, where $A \in \mathbb{R}^{m \times n}$. Then the followings are true

$$\text{Im}(A) = \text{Ker}(A^\top)^\perp \subset \mathbb{R}^m \tag{11a}$$

$$\text{Im}(A^\top) = \text{Ker}(A)^\perp \subset \mathbb{R}^n \tag{11b}$$

Furthermore

$$\text{Im}(A) \otimes \text{Ker}(A^\top) = \mathbb{R}^m \tag{12a}$$

$$\text{Im}(A^\top) \otimes \text{Ker}(A) = \mathbb{R}^n \tag{12b}$$

Remark. If $r = \text{rank}(A)$, than

$$\dim \text{Im}(A) = r, \quad \dim \text{Ker}(A^\top) = m - r \tag{13a}$$

$$\dim \text{Im}(A^\top) = r, \quad \dim \text{Ker}(A) = n - r \tag{13b}$$

Proof. Proof of (11a) as presented in [1]. Let

$$A = (\mathbf{a}_1 \quad \mathbf{a}_2 \quad \dots \quad \mathbf{a}_n) \Rightarrow A^\top = \begin{pmatrix} \mathbf{a}_1^\top \\ \mathbf{a}_2^\top \\ \vdots \\ \mathbf{a}_n^\top \end{pmatrix} \tag{14a}$$

$$\mathbf{x} \in \text{Ker}(A^\top) \Rightarrow A^\top \mathbf{x} = \begin{pmatrix} \mathbf{a}_1^\top \mathbf{x} \\ \mathbf{a}_2^\top \mathbf{x} \\ \vdots \\ \mathbf{a}_n^\top \mathbf{x} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \dots \\ 0 \end{pmatrix} \tag{14b}$$

$$\mathbf{y} \in \text{Im}(A) \Rightarrow \exists \alpha_i \in \mathbb{R} \text{ such that } \mathbf{y} = \sum_{i=1}^n \alpha_i \mathbf{a}_i \tag{14c}$$

Note that \mathbf{x} and \mathbf{y} are arbitrary vector elements of $\text{Ker}(A^\top)$ and $\text{Im}(A)$, respectively. Then we compute the dot product of \mathbf{x} and \mathbf{y} :

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{y}^\top \mathbf{x} = \sum_{i=1}^n \alpha_i \mathbf{a}_i^\top \mathbf{x} = 0, \tag{15}$$

since $\mathbf{a}_i^\top \mathbf{x} = 0$, $\forall i = \overline{1, n}$. Consequently, $\mathbf{x} \perp \mathbf{y}$ for all possible $x \in \text{Ker}(A^\top)$ and $y \in \text{Im}(A)$, which means that the two subspaces are the **orthogonal complement** for each other:

$$\begin{aligned} \text{Im}(A) &= \text{Ker}(A^\top)^\perp \\ \text{Im}(A) \cap \text{Ker}(A^\top) &= \{0\} \end{aligned} \tag{16}$$

Following this idea, we can conclude that the subspaces are linearly independent, therefore,

$$\dim \left(\text{Im}(A) \otimes \text{Ker}(A^\top) \right) = r + (m - r) = m. \tag{17}$$

This can only happend if **direct product** of the two spaces is \mathbb{R}^m , which completes the proof for (12a). □

Proposition 5. (Self-adjoint operator) Let $\mathcal{A} : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $\mathcal{A}(x) = Ax$, where $A \in \mathbb{R}^{n \times n}$ is a symmetric matrix: $A = A^\top$. Than, as a consequence of Theorem 4, we have that

$$\text{Im}(A) = \text{Ker}(A)^\perp \text{ and } \text{Im}(A) \otimes \text{Ker}(A) = \mathbb{R}^n.$$

For more, see [2, Eq. (10.3)].

Proposition 6.

Singular value decomposition (SVD)

 If we make the SVD for matrix $A \in \mathbb{R}^{m \times n}$

$$A = U\Sigma V^\top, \quad (18)$$

where

$$U \in \mathbb{R}^{m \times m} \text{ is unitary: } U^*U = I_m \quad (19a)$$

$$V \in \mathbb{R}^{n \times n} \text{ is unitary: } V^*V = I_n \quad (19b)$$

$$\Sigma \in \mathbb{R}^{m \times n} \text{ eigenvalues in the diagonal.} \quad (19c)$$

After this decomposition, the basis of the four subspaces (12) can be obtained as presented below.

 $\text{Im}(A) :$ the first r columns of U
 $\text{Ker}(A^\top) :$ the last $m - r$ columns of U
 $\text{Im}(A^\top) :$ the first r columns of V
 $\text{Ker}(A) :$ the last $n - r$ columns of V

In short

$$“ A = [\text{Im}(A) \quad \text{Ker}(A^\top)] \Sigma [\text{Im}(A^\top) \quad \text{Ker}(A)]^\top ” \quad (20)$$

B Subspaces of the state space

 Having a strictly proper ($D = 0$) MIMO LTI system:

$$\begin{aligned} \dot{x} &= Ax + By \\ y &= Cx \end{aligned} \quad (21)$$

The state space could be partitioned as follows:

$$X = X_{co} \otimes X_{c\bar{o}} \otimes X_{\bar{c}o} \otimes X_{\bar{c}\bar{o}} \quad (22)$$

 where $X_{..}$ are pairwise orthogonal subspaces of the state space, in other words:

$$\begin{aligned} X_{co} \perp X_{c\bar{o}}, \quad X_{co} \perp X_{\bar{c}o}, \quad X_{co} \perp X_{\bar{c}\bar{o}}, \\ X_{c\bar{o}} \perp X_{\bar{c}o}, \quad X_{c\bar{o}} \perp X_{\bar{c}\bar{o}}, \quad X_{\bar{c}o} \perp X_{\bar{c}\bar{o}}. \end{aligned} \quad (23)$$

B.1 Unobservable subspace $X_{\bar{o}} = \text{Ker}(\mathcal{O}_n)$. Observable subspace $X_o = X_{\bar{o}}^\perp = \text{Im}(\mathcal{O}_n^\top)$.

Lemma 7.

 Linear independence of the first k rows of \mathcal{O}_n

 If $\text{rank}(\mathcal{O}_n) = k \leq n$, then the first k rows of \mathcal{O}_n are linearly independent, and any further rows of it can be expressed as the linear combination of the first k rows.

 Formally: $\forall i \in \mathbb{N} \exists \alpha \in \mathbb{R}^k$, that $CA^{k+i} = \alpha^\top \mathcal{O}_k$, where $\mathcal{O}_k \in \mathbb{R}^{k \times n}$ is defined as $\mathcal{O}_k = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{k-1} \end{pmatrix}$.

Remark. $\mathbb{N} = \{0, 1, 2, \dots\}$, $\mathbb{N}_+ := \mathbb{N} \setminus \{0\}$.

Proof. The proof is given in the following three steps:

- (i) If $k = n$, the set of row vectors (also called as ‘‘covariant vectors’’) C, CA, \dots, CA^{n-1} constitutes a linearly independent (covariant) basis for vector space \mathbb{R}^n , which means that any other row vectors in \mathbb{R}^n can be expressed by their linear combinations, the same as $CA^{n+i}, \forall i \in \mathbb{N}$ can be.
- (ii) Let k be the first natural number, for which there exists $\alpha \in \mathbb{R}^k$ such that $CA^k = \alpha^\top \mathcal{O}_k$. Then

CA^{k+1} can also be expressed by the covariant vectors of \mathcal{O}_k :

$$CA^{k+1} = (CA^k)A = \left(\sum_{j=1}^k \alpha_j CA^{j-1} \right) A = \sum_{j=1}^{k-1} \alpha_j CA^j + \alpha_k \sum_{j=1}^k \alpha_j CA^{j-1} \quad (24)$$

By induction, we have that for every $i \in \mathbb{N}$ there exists $\alpha \in \mathbb{R}^k : CA^{k+i} = \alpha \mathcal{O}_k$.

(iii) As a consequence of (ii), we can state that if $\text{rank}(\mathcal{O}_n) = k < n$, that the first k rows of \mathcal{O}_n are linearly independent (i.e. $\text{rank}(\mathcal{O}_k) = k$). \square

Lemma 8. For every $v \in \text{Im}(\mathcal{O}_n^\top)$, we have that $A^\top v \in \text{Im}(\mathcal{O}_n^\top)$. In this sense, the observable subspace $X_o = \text{Im}(\mathcal{O}_n^\top) = \text{Ker}(\mathcal{O}_n)^\perp \subseteq \mathbb{R}^n$ of the state space $X = \mathbb{R}^n$ is invariant with respect to the linear transformation $\mathcal{A}'(v) = A^\top v$, i.e. $\mathcal{A}'(X_o) = X_o$.

Proof. Let $a(s) = \det(sI - A) = a_0 + a_1s + \dots + a_ns^n$. Due to Cayley-Hamilton theorem, we have that

$$a(A) = 0 \Rightarrow A^n = -\frac{1}{a_n}(a_0I + a_1A + \dots + a_{n-1}A^{n-1}) \quad (25)$$

\square

Proposition 9.

$x(0) \in \text{Ker}(\mathcal{O}_n)$ and $u(t) = 0 \Rightarrow y(t) = 0$

Let $\text{rank}(\mathcal{O}_n) = k < n$. If $x_0 \in \text{Ker}(\mathcal{O}_n)$ and $u \equiv 0$, than $y(t) = 0$ for every $t > 0$, i.e

$$x(t) = e^{At}x_0 \in \text{Ker}(\mathcal{O}_n)$$

In other words, if there is no input signal ($u(t) = 0$) and the initial condition x_0 belongs to the unobservable subspace $\text{Ker}(\mathcal{O}_n)$, than the state response of the system $x(t) = e^{At}x_0$ will remain in this subspace.

Proof. As a consequence of Proposition 7, we have that if $CA^k x_0 = 0$ for $k = \overline{0, n-1}$, than $CA^k x_0 = 0$ holds for every $k \in \mathbb{N}$. If we consider the Taylor expansion of matrix exponent e^{At} , we have:

$$CA^k e^{At}x_0 = \sum_{j=0}^{\infty} \frac{t^k}{k!} \cdot \underbrace{CA^{k+j}x_0}_0 = 0 \quad \forall k = \overline{0, n-1} \Rightarrow \mathcal{O}_n e^{At}x_0 = 0 \Leftrightarrow e^{At}x_0 \in \text{Ker}(\mathcal{O}_n) \quad (26)$$

Consequently, for a given unobservable state space model (A,B,C,D) if we start the system from the unobservable subspace $x(0) \in \text{Ker}(\mathcal{O}_n)$ and having a zero input ($u \equiv 0$) the output will be zero $y(t) = 0$, for every $t > 0$. \square

Proposition 10.

Same output for all initial state of an unobservable class

Let us denote $v_1, \dots, v_{n-k} \in \mathbb{R}^n$, $k < n$ the basis vectors of the null space of \mathcal{O}_n :

$$\text{Ker}(\mathcal{O}_n) = \{ \alpha_1 v_1 + \dots + \alpha_{n-k} v_{n-k} = \alpha^\top N \mid \alpha \in \mathbb{R}^{n-k} \}, \quad \text{where } N := (v_1 \dots v_{n-k}) \in \mathbb{R}^{n \times (n-k)}$$

Matrix N is called an *annihilator* of \mathcal{O}_n , since $\mathcal{O}_n N = 0_{n \times (n-k)}$. Now we introduce the following notations:

$$x_0 + \text{Ker}(\mathcal{O}_n) := \{ x_0 + \alpha^\top N \mid \alpha \in \mathbb{R}^{n-k} \} \quad (27)$$

From any initial condition $x(0) \in x_0 + \text{Ker}(\mathcal{O}_n)$ and for a given input $u(t)$, the system will produce the same output $y(t)$.

Proof. The explicit solution of the state space model is

$$y(t) = Ce^{At}x(0) + C \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau \quad (28)$$

Considering an initial condition $x(0) = x_0 + \alpha^\top N \in x_0 + \text{Ker}(\mathcal{O}_n)$ with an arbitrary $\alpha \in \mathbb{R}^{n-k}$, and keeping in mind, that $\alpha^\top N \in \text{Ker}(\mathcal{O}_n)$ (i.e. $CA^i \alpha^\top N = 0$ for all $i \in \mathbb{N}$) we obtain:

$$y(t) = Ce^{At}(x_0 + \alpha^\top N) + C \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau = Ce^{At}x_0 + C \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau \quad (29)$$

Finally, we can observe that the expression for $y(t)$ does not depend on α . It depends only on the input $u(t)$ and on x_0 , furthermore, for each x_0 we obtain different outputs, x_0 defines the unobservability class, that the system is actually in. If we can find a particular solution $x(t)$ for the (under-determined) linear equation system

$$\mathcal{Y}(t) = \mathcal{O}_n x(t) + \mathcal{TU}(t) \quad [\text{lec_03.pdf, pg. 10/31}] \quad (30)$$

we can determine the actual unobservability class of the system, but we have no further informations about the state vector itself. \square

Remark. Set $x_0 + \text{Ker}(\mathcal{O}_n)$ is not a subspace of \mathbb{R}^n , since many properties of the vector space broke (eg. does not have a unity element), however, it is a k dimensional manifold (sokaság) in vector space \mathbb{R}^n .

B.2 Controllable subspace $X_c = \text{Im}(\mathcal{C}_n)$. Uncontrollable subspace $X_{\bar{c}} = X_c^\perp = \text{Ker}(\mathcal{C}_n^\top)$.

Lemma 11. If (A, B, C) is not controllable $\text{rank}(\mathcal{C}_n) = k < n$, the first k columns of \mathcal{C}_n are linearly independent.

Proof. Same as Lemma 7. \square

Lemma 12. For every $v \in \text{Im}(\mathcal{C}_n)$, vector $Av \in \text{Im}(\mathcal{C}_n)$. In this sense, the controllable subspace $X_c = \text{Im}(\mathcal{C}_n) \subseteq \mathbb{R}^n$ of the state space $X = \mathbb{R}^n$ is invariant with respect to the linear transformation $\mathcal{A}(v) = Av$, i.e. $\mathcal{A}(X_c) = X_c$.

Proof. Let $v \in X_c = \text{span}\langle B, AB, \dots, A^{n-1}B \rangle$, therefore, there exist real values $\alpha_1, \dots, \alpha_n \in \mathbb{R}$, such that

$$v = \sum_{i=1}^n \alpha_i A^{i-1} B \Rightarrow Av = \sum_{i=1}^n \alpha_i A^i B. \quad (31)$$

It is obvious that $A^i B \in X_c$ for all $i = \overline{1, n-1}$, furthermore, due to Lemma 11, $A^n B$ can be expressed as the linear combination of vectors $A^{i-1} B, B, i = \overline{1, n}$. Finally, we have that $Av \in X_c$. \square

Proposition 13.

$$x_0 \in \text{Im}(\mathcal{C}_n) \Rightarrow x(t) \in \text{Im}(\mathcal{C}_n)$$

If the initial condition $x(0) = x_0$ belongs to the controllable subspace of the state space, than the solution $x(t)$ will also belong to it. Formally:

$$x_0 \in \text{Im}(\mathcal{C}_n) \Rightarrow x(t) \in \text{Im}(\mathcal{C}_n) \forall t \geq 0. \quad (32)$$

If the initial condition is not an element of $\text{Im}(\mathcal{C}_n)$, but the system is stable, than the trajectory will tend exponentially to the controllable subspace of the state space, i.e.

$$A \prec 0 \Rightarrow x(t) \rightarrow \text{Im}(\mathcal{C}_n) \quad (33)$$

Proof. If $x_0 \in \text{Im}(\mathcal{C}_n) = X_c$, than

$$x(t) = e^{At} x_0 + \int_0^t e^{A(t-\tau)} B u(\tau) d\tau = \sum_{k=0}^{\infty} \frac{t^k}{k!} \underbrace{A^k x_0}_{\in X_c} + \int_0^t \sum_{k=0}^{\infty} \frac{(t-\tau)^k}{k!} \underbrace{A^k B}_{\in X_c} u(\tau) d\tau \in X_c. \quad (34)$$

If $x_0 \notin X_c$ but $A \prec 0$ (is negative definite), than

$$x(t) = \underbrace{e^{At} x_0}_{\rightarrow 0} + \underbrace{\int_0^t e^{A(t-\tau)} B u(\tau) d\tau}_{\in X_c} \rightarrow X_c. \quad (35)$$

So, the solution tends to the controllable subspace. \square

Theorem 14. (Control the system to a given state) If the system is controllable, there exists an input

$$u(t) = -B^\top e^{A^\top(t-t_1)} P^{-1}(t_1) (e^{At_1} x_0 - x_1), \text{ where } P(t) = \int_0^t e^{A\tau} B B^\top e^{A^\top \tau} d\tau, \quad t \in [0, t_1], \quad (36)$$

which leads the system from $x(0)$ to $x(t_1) = x_1$ in a finite time $t_1 < \infty$.

Proof. A proof for it can be found in [3, Theorem 2.21]. \square

B.3 Controllability staircase form

Proposition 15.

Controllability staircase form

We construct the following transformation matrix $T^{-1} = S = (v_1, \dots, v_k, w_{k+1}, \dots, w_n)$, where $[v] = [v_1, \dots, v_k]$ is the orthonormal (ON) basis of $X_c = \text{Im}(\mathcal{C}_n)$ and $[w] = [w_{k+1}, \dots, w_n]$ is the ON basis of $X_{\bar{c}} = \text{Im}(\mathcal{C}_n)^\perp = \text{Ker}(\mathcal{C}_n^\top)$. Then the transformed matrices will have the form:

$$\bar{A} = TAT^{-1} = \begin{pmatrix} \bar{A}_{11} & \bar{A}_{12} \\ 0_{(n-k) \times k} & \bar{A}_{22} \end{pmatrix} \quad (37a) \quad \bar{B} = TB = \begin{pmatrix} \bar{B}_1 \\ 0_{(n-k) \times 1} \end{pmatrix} \quad (37b)$$

Using SVD: $\mathcal{C}_n = U_c \Sigma_c V_c^\top$, $S := U_c$

Proof. (For simplicity, only for SISO) Since X_c and $X_{\bar{c}}$ are orthogonal complement of each other (i.e. $X_c \otimes X_{\bar{c}} = \mathbb{R}^n$), $[v, w]$ is an ON basis of \mathbb{R}^n . In other words: S is an orthogonal matrix with the well-known properties:

$$S^\top S = I_n \quad \Rightarrow \quad S^{-1} = S^\top = \begin{pmatrix} V^\top \\ W^\top \end{pmatrix}, \quad \text{where } V = (v_1, \dots, v_k) \text{ and } W = (w_{k+1}, \dots, w_n) \quad (38)$$

Furthermore, $V^\top W = 0_{k \times (n-k)}$ and $W^\top V = 0_{(n-k) \times k}$ (39). Then the transformed matrix \bar{A} will be:

$$\bar{A} = TAT^{-1} = S^\top AS = \begin{pmatrix} V^\top \\ W^\top \end{pmatrix} A \begin{pmatrix} V & W \end{pmatrix} = \begin{pmatrix} V^\top AV & V^\top AW \\ W^\top AV & W^\top AW \end{pmatrix}. \quad (40)$$

The columns of V are elements of X_c , therefore, the columns of AV are also elements of X_c . The columns of W are the basis vectors of $X_{\bar{c}} = X_c^\perp$, therefore, $W^\top AV = 0_{(n-k) \times k}$. The transformed matrix \bar{B} will be:

$$\bar{B} = TB = S^\top B = \begin{pmatrix} V^\top \\ W^\top \end{pmatrix} B = \begin{pmatrix} V^\top B \\ W^\top B \end{pmatrix}. \quad (41)$$

Since $B \in X_c$, $w_j \in X_c^\perp$, $W^\top B = 0_{(n-k) \times 1}$, $j = \overline{k+1, n}$. \square

B.4 Observability staircase form

Proposition 16.

Observability staircase form

We construct the following transformation matrix $T^{-1} = S = (v_1, \dots, v_k, w_{k+1}, \dots, w_n)$, where $[v] = [v_1, \dots, v_k]$ is the orthonormal (ON) basis of $X_o = \text{Ker}(\mathcal{O}_n)^\perp = \text{Im}(\mathcal{O}_n^\top)$ and $[w] = [w_{k+1}, \dots, w_n]$ is the ON basis of $X_{\bar{o}} = \text{Ker}(\mathcal{O}_n)$. Then the transformed matrices will have the form:

$$\bar{A} = TAT^{-1} = \begin{pmatrix} \bar{A}_{11} & 0_{k \times (n-k)} \\ \bar{A}_{21} & \bar{A}_{22} \end{pmatrix} \quad (42a) \quad \bar{C} = CT^{-1} = (\bar{C}_1 \quad 0_{1 \times (n-k)}) \quad (42b)$$

Using SVD: $\mathcal{O}_n = U_o \Sigma_o V_o^\top$, $S := V_o$

Proof. (For simplicity, only for SISO) Since X_o and $X_{\bar{o}}$ are orthogonal complement of each other (i.e. $X_o \otimes X_{\bar{o}} = \mathbb{R}^n$), $[v, w]$ is an ON basis of \mathbb{R}^n . In other words: S is an orthogonal matrix with the well-known properties:

$$S^\top S = I_n \quad \Rightarrow \quad S^{-1} = S^\top = \begin{pmatrix} V^\top \\ W^\top \end{pmatrix}, \quad \text{where } V = (v_1, \dots, v_k) \text{ and } W = (w_{k+1}, \dots, w_n) \quad (43)$$

Furthermore, $V^\top W = 0_{k \times (n-k)}$ and $W^\top V = 0_{(n-k) \times k}$ (44). The transformed matrix \bar{A} will be:

$$\bar{A} = TAT^{-1} = S^\top AS = \begin{pmatrix} V^\top \\ W^\top \end{pmatrix} A \begin{pmatrix} V & W \end{pmatrix} = \begin{pmatrix} V^\top AV & V^\top AW \\ W^\top AV & W^\top AW \end{pmatrix}. \quad (45)$$

The columns of V are elements of X_o , therefore, the columns of $A^\top V$ are also elements of X_o . The columns of W are the basis vectors of $X_{\bar{c}} = X_c^\perp$, therefore, $(A^\top V)^\top W = V^\top AW = 0_{k \times (n-k)}$. The transformed matrix \bar{C} will be:

$$\bar{C} = CT^{-1} = CS = C \begin{pmatrix} V & W \end{pmatrix} = (CV \quad CW). \quad (46)$$

Since $C^\top \in X_o$, $w_j \in X_o^\perp$, $CW^\top = 0_{1 \times (n-k)}$, $j = \overline{k+1, n}$. \square

Proposition 17. If (A, C) has unobservable mode (i.e. is unobservable), there exists $x \in \mathbb{R}^n$, such that $Ax = \lambda x$ and $Cx = 0$. Consequently, λ is a “decoupling zero” of (A, B, C, D) , since

$$M = \begin{pmatrix} A - \lambda I & B \\ C & 0 \end{pmatrix} \text{ is singular,} \quad (47)$$

namely there exists $\xi = \begin{pmatrix} x \\ 0 \end{pmatrix} \neq 0$ such that $M\xi = 0$. Or in other words, the kernel space of M is not empty, meaning that M is singular.

Proposition 18. The input decoupling zeros are equal to the eigenvalues of the uncontrollable subsystem.

Proof. We assume that (A, B) is uncontrollable:

$$\mathcal{C}_n = (B \quad AB \quad \dots \quad A^{n-1}B) \in \mathbb{R}^{n \times mn} \quad (48)$$

is rank deficient, that implies a nonempty kernel space $\text{Ker}(\mathcal{C}_n^\top) \subset \mathbb{R}^n$, namely, there exists $x \in \mathbb{R}^n$ such that $\mathcal{C}_n^\top x = 0$. Alternatively, we have that

$$\begin{cases} B^\top x = 0 \\ B^\top A^\top x = 0 \\ \dots \\ B^\top (A^\top)^{n-1} x = 0 \end{cases} \quad (49)$$

\square

B.5 Kalman decomposition

We produce a controllability staircase form decomposition on the system, than on both subsystems (controllable and uncontrollable) we produce an observability staircase form decomposition.