

Computer controlled systems

Lecture 2

version: 2020.09.11. – 10:58:32

Problem types

- Convolution of two casual signals
- Solution of initial value probles using Laplace transformation, eg. $\ddot{y} + a\dot{y} + by = e^{-at}$, $y(0)$, $\dot{y}(0)$, $y(t) = ?$
- Partial fractional decomposition
- Determine the transfer function of $\ddot{y} + a\dot{y} + by = u(t)$.
- Solution of initial values problem in the state-space form using Laplace tr, eg. $\dot{x} = Ax$, $x(0)$
- Compute the transfer function ($H(s)$) of the state space model ($\dot{x} = Ax + Bu$, $y = Cx$)
- Solution of the state space model, given both the input and the initial values – impulse response ($h(t)$), response to the unit step function

1 Causal convolution

Definition 1. A signal $f(t)$ is called causal if $f(t < 0) = 0$.

Definition 2. (Causal convolution) The convolution of two causal signals $f(t)$ and $g(t)$ reduces to:

$$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau)g(t - \tau)d\tau = \int_0^{\infty} f(\tau)g(t - \tau)d\tau = \int_0^t f(\tau)g(t - \tau)d\tau$$

2 Laplace transformation

Definition 3. The Laplace transform of a signal $f(t)$ is denoted by $F(s)$, $s \in \mathbb{C}$ and it is defined as follows:

$$F(s) = \mathcal{L}\{f(t), s\} = \int_0^{\infty} f(t)e^{-st} dt \quad (1)$$

Due to the fact that the integral operator is linear, the Laplace transformation is inherently linear, i.e. it preserves the order of addition and scaling operations.

Based on the properties of the integral the laplace transform is a linear mapping.

2.1 Rules

1. Convolution in time domain: $\mathcal{L}\{(f * g)(t), s\} = F(s)G(s),$

where $F(s) = \mathcal{L}\{f(t), s\}$, $G(s) = \mathcal{L}\{g(t), s\}$, $(f * g)(t) = \int_0^t f(\tau)g(t - \tau)d\tau$. Derivation:

$$\begin{aligned} \mathcal{L}\{(f * g)(t), s\} &= \int_0^\infty \int_0^t f(\tau)g(t - \tau)d\tau e^{-st}dt = \int_0^\infty \int_0^\infty f(\tau)g(t - \tau)e^{-st}dt d\tau \\ &= \int_0^\infty \int_0^\infty g(t - \tau)e^{-s(t - \tau)}dt f(\tau)e^{-s\tau}d\tau = \int_0^\infty \int_{-\tau}^\infty g(\vartheta)e^{-s\vartheta}d\vartheta f(\tau)e^{-s\tau}d\tau \\ &= \int_0^\infty g(\vartheta)e^{-s\vartheta}d\vartheta \int_0^\infty f(\tau)e^{-s\tau}d\tau = \mathcal{L}\{f(t), s\} \mathcal{L}\{g(t), s\} \end{aligned} \quad (2)$$

We will deal with functions for which $f(t) = g(t) = 0$ for all $t < 0$, hence

$$\int_0^t f(\tau)g(t - \tau)d\tau = \int_0^\infty f(\tau)g(t - \tau)d\tau \quad \text{mivel } g(t - \tau) = 0 \text{ bármely } \tau > t \quad (3)$$

It was also used during the above derivation (change of variables: $\vartheta = t - \tau$):

$$\int_0^\infty g(t - \tau)e^{-s(t - \tau)}dt = \int_{-\tau}^\infty g(\vartheta)e^{-s\vartheta}d\vartheta = \int_0^\infty g(\vartheta)e^{-s\vartheta}d\vartheta \quad \text{since } g(t < 0) = 0 \quad (4)$$

2. Time derivative:

$$\mathcal{L}\{\dot{y}(t), s\} = sY(s) - y(0), \quad \text{ahol } Y(s) = \mathcal{L}\{y(t), s\}. \text{ Derivation:}$$

$$\int_0^\infty \dot{y}(t)e^{-st}dt = y(t)e^{-st} \Big|_0^\infty - (-s) \int_0^\infty y(t)e^{-st}dt = -y(0) + s\mathcal{L}\{y(t), s\} \quad (5)$$

3. Second derivative according to the time variable:

$$\mathcal{L}\{\ddot{y}(t), s\} = s^2Y(s) - \dot{y}(0) - sy(0). \quad \text{Derivation:}$$

$$\mathcal{L}\{\ddot{y}(t), s\} = s\mathcal{L}\{\dot{y}(t), s\} - \dot{y}(0) = s^2Y(s) - sy(0) - \dot{y}(0) \quad (6)$$

4. Second derivative according to the time variable:

$$\mathcal{L}\{\ddot{y}(t), s\} = s^3Y(s) - \ddot{y}(0) - s\dot{y}(0) - s^2y(0).$$

2.2 Limit theorems

1. Initial value theorem. If $e^{-st}y(t)$ is Lebesgue integrable on $t \in \mathbb{R}_+$, then $y(0) = \lim_{s \rightarrow \infty} sY(s)$

Proof. Let us take the limit of both the left and right sides of the rule of derivation $s \rightarrow \infty$:

$$\lim_{s \rightarrow \infty} \int_0^\infty e^{-st}\dot{y}(t)dt \stackrel{(*)}{=} \int_0^\infty \underbrace{\lim_{s \rightarrow \infty} e^{-st}\dot{y}(t)}_{=0}dt = \lim_{s \rightarrow \infty} (sY(s) - y(0)) \Rightarrow y(0) = \lim_{s \rightarrow \infty} sY(s) \quad (7)$$

(*)Due to Lebesgue's dominated convergence theorem, since $e^{-st}\dot{y}(t)$ is (by assumption) Lebesgue integrable on \mathbb{R}_+ .

2. Limit value theorem. If $e^{-st}y(t)$ is Lebesgue integrable on $t \in \mathbb{R}_+$, then $y(\infty) = \lim_{s \rightarrow 0} sY(s)$

Proof. Let us take the derivation rule

$$\int_0^\infty \dot{y}(t)e^{-st}dt = sY(s) - y(0) \quad (8)$$

and consider the limit of both sides $s \rightarrow 0$:

$$\lim_{s \rightarrow 0} \int_0^{\infty} e^{-st} \underbrace{\dot{y}(t) dt}_{dy(t)} \stackrel{(*)}{=} \int_0^{\infty} \underbrace{\lim_{s \rightarrow 0} e^{-st}}_{\rightarrow 1} \underbrace{\dot{y}(t) dt}_{dy(t)} = \lim_{s \rightarrow 0} (sY(s) - y(0)) \quad (9)$$

$$\int_0^{\infty} dy(t) = \lim_{s \rightarrow 0} sY(s) - y(0) \quad (10)$$

$$y(\infty) - y(0) = \lim_{s \rightarrow 0} sY(s) - y(0) \Rightarrow y(\infty) = \lim_{s \rightarrow 0} sY(s) \quad (11)$$

(*)Due to Lebesgue's dominated convergence theorem, since $e^{-st}\dot{y}(t)$ is (by assumption) Lebesgue integrable on \mathbb{R}_+ .

Counterexample if $e^{-st}y(t)$ is not absolute integrable for all $\text{Re}\{s\} > 0$. Let $y(t) = e^t$. Its Laplace transformation is $Y(s) = \frac{1}{s-1}$. $\lim_{s \rightarrow 0} \frac{s}{s-1} = 0$, however $y(t) \rightarrow \infty$.

2.3 Laplace transformation of some important functions

$$1. \quad \mathcal{L}\{\delta(t), s\} = 1 \quad \text{derivation: } \int_0^{\infty} \delta(t)e^{-st} dt = \lim_{T \rightarrow 0} \frac{1}{T} \int_0^T e^{-st} dt = \frac{1}{s} \lim_{T \rightarrow 0} \underbrace{\frac{1 - e^{-sT}}{T}}_{\text{L'Hospital: } s} = 1$$

Note that in this derivation Dirac's function is estimated as follows:

$$\delta(t) = \lim_{T \rightarrow 0} \begin{cases} \frac{1}{T} & t \in [0, T] \\ 0 & \text{otherwise} \end{cases} \quad (12)$$

$$2. \quad \mathcal{L}\{1(t), s\} = \frac{1}{s} \quad \text{derivation: } \int_0^{\infty} 1(t)e^{-st} dt = \left[\frac{e^{-st}}{-s} \right]_0^{\infty} = 0 - \left(-\frac{1}{s}\right) = \frac{1}{s},$$

where $1(t)$ unit step function, for this the $u(t)$ notation is also commonly used, but in this course $u(t)$ denotes the input of the system.

$$3. \quad \mathcal{L}\{t \cdot 1(t), s\} = \frac{1}{s^2} \quad (\text{unit step velocity function})$$

$$4. \quad \mathcal{L}\{e^{-at}, s\} = \frac{1}{s+a}, \quad \text{it is the most commonly used when trying to determine inverse Laplace transformations of complex rational functions.}$$

$$5. \quad \mathcal{L}\{e^{-t/T}, s\} = \frac{1}{s+1/T} = \frac{T}{1+sT}, \quad \text{it is another form of the previous case.}$$

$$\text{Derivation: } \int_0^{\infty} e^{-t/T} e^{-st} dt = \int_0^{\infty} e^{-(s+1/T)t} dt \left[\frac{e^{-(s+1/T)t}}{-(s+1/T)} \right]_0^{\infty} = \frac{1}{s+1/T} = \frac{T}{1+sT}.$$

Pole-zero from: $\frac{1}{s+1/T}$

Time-constant form: $\frac{T}{1+sT}$

$$6. \quad \mathcal{L}\{1 - e^{-t/T}, s\} = \frac{1}{s(1+sT)} \quad (\text{time-constant form})$$

$$7. \quad \mathcal{L}\left\{ \frac{1}{T_1 - T_2} (e^{-t/T_1} - e^{-t/T_2}), s \right\} = \frac{1}{(1+sT_1)(1+sT_2)} \quad (\text{time-constant form})$$

$$8. \quad \mathcal{L}\{e^{at} \sin(bt), s\} = \frac{b}{(s-a)^2 + b^2}$$

$$9. \quad \mathcal{L}\{e^{at} \cos(bt), s\} = \frac{s-a}{(s-a)^2 + b^2}$$

2.4 Inverse Laplace transform

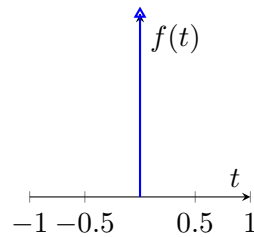
Definition 4. The inverse Laplace transformation of the complex signal $F(s)$ is given by the following complex line integral:

$$f(t) = \mathfrak{L}^{-1}\{F(s)\} = \frac{1}{2\pi j} \lim_{T \rightarrow \infty} \int_{c-jT}^{c+jT} F(s)e^{st} ds \quad (13)$$

where $c \in \mathbb{R}$ is greater than the real parts of $F(s)$'s singularities.

2.5 Input, system response

1. Dirac impulse



$$f_\tau(t) = \begin{cases} \frac{1}{\tau} & \text{if } 0 \leq t < \tau \\ 0 & \text{otherwise} \end{cases} \quad \delta(t) = \lim_{\tau \rightarrow 0^+} f_\tau(t)$$

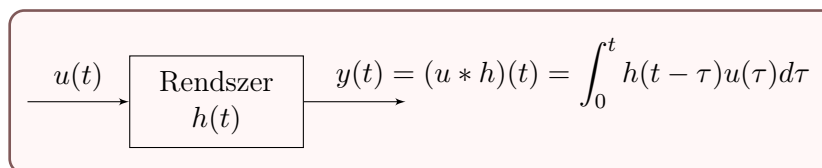
2. the output (response) of the system to the Dirac impulse (impulse response): $h(t)$

E.g.. if I strike on a trapdoor ($\delta(t)$), then it will dampedly oscillate($h(t)$).



Convolutional time-invariance: $\delta(t - \tau)$, $h(t - \tau)$.

3. The system response to $u(t)$ (transfer function): Causal convolution



Example 1.

Let us compute the convolution of $f(t) = t$ and $g(t) = t^2$:

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau = \int_0^t g(\tau)f(t-\tau)d\tau = \int_0^t (t-\tau)\tau^2 d\tau = \int_0^t t\tau^2 - \tau^3 d\tau = \left[\frac{t\tau^3}{3} - \frac{\tau^4}{4} \right]_0^t = \frac{t^4}{12} \quad (14)$$

3 Laplace transformation to solve initial value problems

Example 2.

Constant coefficient second order linear differential equation

Solve the following initial value problem:

$$\ddot{y} - 2\dot{y} + 5y = -8e^{-t} \quad y(0) = 2 \quad \dot{y}(0) = 12$$

One can compute the Laplace transform as follows (elementwise).

$$\mathfrak{L}\{\ddot{y}\} - 2\mathfrak{L}\{\dot{y}\} + 5\mathfrak{L}\{y\} = -\frac{8}{s+1} \quad (15)$$

Laplace transform in the case of derivated function: $\mathfrak{L}\{\dot{y}\} = sY(s) - y(0) = sY(s) - 2$. and the second derivative: $\mathfrak{L}\{\ddot{y}\} = s^2Y(s) - sy(0) - \dot{y}(0) = s^2Y(s) - 2s - 12$. such a way the equation (15) has the following form:

$$(s^2Y(s) - 2s - 12) - 2(sY(s) - 2) + 5Y(s) = -\frac{8}{s+1} \quad (16)$$

expressing $Y(s)$ we get:

$$Y(s) = \frac{2s^2 + 10s}{(s^2 - 2s + 5)(s + 1)} \xrightarrow{\mathfrak{L}^{-1}} y(t) = ? \quad (17)$$

Partial fractional decomposition (“brute force” approach):

$$Y(s) = \frac{2s^2 + 10s}{(s^2 - 2s + 5)(s + 1)} = \frac{As + B}{s^2 - 2s + 5} + \frac{C}{s + 1} \quad (18)$$

We multiply by the denominators:

$$2s^2 + 10s = As^2 + (A + B)s + B + Cs^2 - 2Cs + 5C \quad (19)$$

$$2s^2 + 10s = (A + C)s^2 + (A + B - 2C)s + (B + 5C) \quad (20)$$

The coefficients of the monomials in s should be the same:

$$\begin{cases} A + C = 2 \\ A + B - 2C = 10 \\ B + 5C = 0 \end{cases} \Rightarrow \begin{cases} A = 2 - C \\ 2 - C - 5C - 2C = 10 \\ B = -5C \end{cases} \Rightarrow \begin{cases} A = 3 \\ B = 5 \\ C = -1 \end{cases} \quad (21)$$

(Almost) finally:

$$Y(s) = \frac{3s + 5}{s^2 - 2s + 5} - \frac{1}{s + 1} \quad (22)$$

The first term is a bit tricky:

$$\frac{3s + 5}{s^2 - 2s + 5} = \frac{3s + 5}{(s - 1)^2 + 2^2} = 3 \cdot \frac{s - 1}{(s - 1)^2 + 2^2} + 4 \cdot \frac{2}{(s - 1)^2 + 2^2} \quad (23)$$

Finally: $y(t) = 3e^t \cos(2t) + 4e^t \sin(2t) - e^{-t}$.

Example 3.

Partial fraction decomposition

Tel us solve the following initial value problem:

$$\ddot{y} + 7\dot{y} + 14y + 8y = 0 \quad y(0) = 0 \quad \dot{y}(0) = 0 \quad \ddot{y}(0) = 2$$

The Laplace transform is the following:

$$Y(s) = \frac{2}{(s + 1)(s + 2)(s + 4)} = \frac{C_1}{s + 1} + \frac{C_2}{s + 2} + \frac{C_3}{s + 4} \xrightarrow{\mathfrak{L}^{-1}} y(t) = C_1e^{-t} + C_2e^{-2t} + C_3e^{-4t}$$

Note that the roots of the denominator have single multiplicity, then the following formula can be applied:

$$C_i = \lim_{s \rightarrow \alpha_i} (s - \alpha_i)Y(s), \text{ ahol } \alpha_i \text{ az } \frac{C_i}{s + \alpha_i} \text{ gyöke}$$

$$C_1 = \lim_{s \rightarrow -1} (s + 1)Y(s) = \frac{2}{(s + 2)(s + 4)} \Big|_{s=-1} = \frac{2}{3}$$

$$C_2 = \lim_{s \rightarrow -2} (s + 2)Y(s) = \frac{2}{(s + 1)(s + 4)} \Big|_{s=-2} = -1$$

$$C_3 = \lim_{s \rightarrow -4} (s + 4)Y(s) = \frac{2}{(s + 1)(s + 2)} \Big|_{s=-4} = \frac{1}{3}$$

$$Y(s) = \frac{\frac{2}{3}}{s + 1} + \frac{-1}{s + 2} + \frac{\frac{1}{3}}{s + 4} \quad (24)$$

The final solution is:

$$y(t) = \frac{2}{3}e^{-t} - e^{-2t} + \frac{1}{3}e^{-4t} \quad (25)$$

Matlab 1. Inverse Laplace transform

`partfrac, ilaplace, residue, poly2sym, sym2poly`

Here we present, how Matlab can be used to compute automated partial fractional decomposition and inverse Laplace transformation.

Example 4. Consider the following problem:

$$Y(s) = \frac{2s^2 + 10s}{(s^2 - 2s + 5)(s + 1)} = \frac{3s + 5}{s^2 - 2s + 5} - \frac{1}{s + 1} \Rightarrow y(t) = 3e^t \left(\cos(2t) + \frac{4 \sin(2t)}{3} \right) - e^{-t}$$

Solve it using Matlab's [Symbolic Math Toolbox](#):

```
>> syms s % Define symbolic variable 's'
>> Y = partfrac( (2*s^2 + 10*s) / ((s+1) * (s^2 - 2*s + 5)) )
Y =
(3*s + 5)/(s^2 - 2*s + 5) - 1/(s + 1)
>> ilaplace(Y)
ans =
3*exp(t)*(cos(2*t) + (4*sin(2*t))/3) - exp(-t)
```

Polynomial multiplication using Matlab Symbolic Math Toolbox:

```
>> Y = expand((s+1) * (s^2 - 2*s + 5))
Y =
s^3 - s^2 + 3*s + 5
```

```
>> B = [2 10 0];
```

$s^2 + 10s$ numerically: [2 10 0]

```
>> A = sym2poly(Y)
A =
```

```
1 -1 3 5
```

```
>> [r,p,k] = residue(B,A)
r =
```

```
1.5 - 2i
1.5 + 2i
-1 + 0i
```

```
p =
```

```
1 + 2i
1 - 2i
-1 + 0i
```

```
k =
```

```
[]
```

Example 5. Consider the following problem:

$$Y(s) = \frac{B(s)}{A(s)} = \sum_i \frac{r_i}{s - p_i} + K(s) = -\frac{1}{s + 1} + \frac{1.5 - 2j}{s - 1 - 2j} + \frac{1.5 + 2j}{s - 1 + 2j} \quad (26)$$

```
>> Y = sum(r ./ (s - p)) + poly2sym(k)
Y =
- 1/(s + 1) + (3/2 - 2i)/(s - 1 - 2i) + (3/2 + 2i)/(s - 1 + 2i)
>> latex(Y)
```

```

ans =
- \frac{1}{s + 1} + \frac{\frac{3}{2} - 2i}{s - 1 - 2i} + \frac{\frac{3}{2} + 2i}{s - 1 + 2i}, [...]
>> ilaplace(Y)
ans =
- exp(-t) + exp(t*(1 - 2i))*(3/2 + 2i) + exp(t*(1 + 2i))*(3/2 - 2i)
>> Y = simplify(Y)
ans =
(2*s*(s + 5))/(s^3 - s^2 + 3*s + 5)
>> ilaplace(Y)
ans =
3*exp(t)*(cos(2*t) + (4*sin(2*t))/3) - exp(-t)

```

$$y(t) = -e^{-t} + e^{t(1-2j)} \left(\frac{3}{2} + 2j \right) e^{t(1+2j)} \left(\frac{3}{2} - 2j \right) = 3e^t \left(\cos(2t) + \frac{4 \sin(2t)}{3} \right) - e^{-t} \quad (27)$$

Example 6.

Constant coefficient linear differential equation system

$$\begin{aligned} \dot{x}_1 &= 2x_1 + 3x_2 \\ \dot{x}_2 &= 2x_1 + x_2 \end{aligned} \quad \rightarrow \quad \dot{x} = Ax \quad A = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} \quad x_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Solution: $x(t) = e^{At}x_0$, $e^{At} = Se^{Dt}S^{-1} = \mathfrak{L}^{-1}\{(sI - A)^{-1}\}$.

from the eigenvalue-eigenvector decomposition of the first equation (previous practice). moreover in 1 dimension:

$$e^{at} = \mathfrak{L}^{-1}\{(s - a)^{-1}\} = \mathfrak{L}^{-1}\left\{\frac{1}{s - a}\right\} \quad (28)$$

Both of the expressions can be used. In this case the second:

$$\det(sI - A) = \begin{vmatrix} s - 2 & -3 \\ -2 & s - 1 \end{vmatrix} = (s - 2)(s - 1) - 6 = s^2 - 3s - 4 = (s - 4)(s + 1) \quad (29)$$

$$(sI - A)^{-1} = \frac{1}{(s - 4)(s + 1)} \begin{pmatrix} s - 1 & 3 \\ 2 & s - 2 \end{pmatrix}$$

According to the linearity of the Laplace transform:

$$e^{At}x_0 = \mathfrak{L}^{-1}\left\{\begin{pmatrix} \frac{3}{(s - 4)(s + 1)} \\ \frac{s - 2}{(s - 4)(s + 1)} \end{pmatrix}\right\}$$

Partial fraction decomposition:

$$\frac{3}{(s - 4)(s + 1)} = \frac{3}{5} \frac{(s + 1) - (s - 4)}{(s - 4)(s + 1)} = \frac{0.6}{s - 4} - \frac{0.6}{s + 1} \quad (30)$$

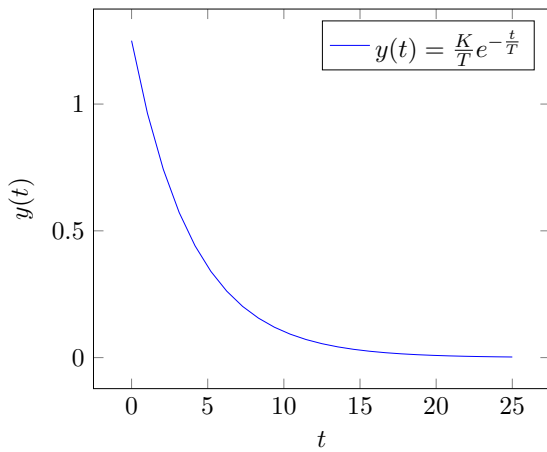
Using a simpler method:

$$\frac{s - 2}{s^2 - 3s - 4} = \frac{C_3}{s + 1} + \frac{C_4}{s - 4} \rightarrow C_3 = 0.6 \quad C_4 = 0.4 \quad (31)$$

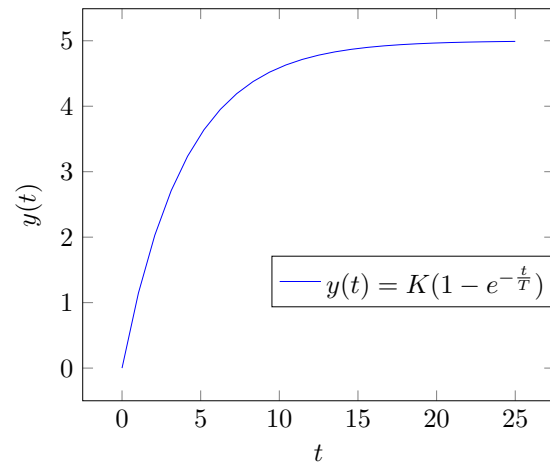
Finally:

$$x(t) = \mathfrak{L}^{-1}\left\{\begin{pmatrix} \frac{-0.6}{s + 1} + \frac{0.6}{s - 4} \\ \frac{0.6}{s + 1} + \frac{0.4}{s - 4} \end{pmatrix}\right\} = \begin{pmatrix} -0.6e^{-t} + 0.6e^{4t} \\ 0.6e^{-t} + 0.4e^{4t} \end{pmatrix} \quad (32)$$

Applying the second formula: $e^{At} = Se^{Dt}S^{-1}$, the decomposition is not required, but the eigenvalues and eigenvectors are necessary.



(a) Response to the Dirac impulse



(b) Response to the unit step impulse

Example 7. Applying Laplace transform

The differential equation system describing the system: $T\dot{y} + y = Ku(t)$ $y(0) = 0$

Let us determine the system's response in the following cases:

1. $u(t) = \delta(t)$
2. $u(t) = 1(t)$

The Laplace transform of the system: $TsY(s) + Y(s) = KU(s)$, where $T \in \mathbb{R}$ and $K \in \mathbb{R}$ parameters depending on the system. Impulse response function:

$$H(s) = \frac{Y(s)}{U(s)} = \frac{K}{1 + Ts} \quad (34)$$

System's response: $Y(s) = \frac{K}{1 + Ts}U(s)$

1. impulse response

$$u(t) = \delta(t) \xrightarrow{\mathcal{L}} U(s) = 1$$

$$Y(s) = K \frac{1}{1 + Ts} \xrightarrow{\mathcal{L}^{-1}} y(t) = \mathcal{L}^{-1}\{Y(s)\} = \frac{K}{T} \mathcal{L}^{-1}\left\{\frac{1}{s + \frac{1}{T}}\right\} = \frac{K}{T} e^{-t/T}$$

2. transfer function (response to the unit step function)

$$u(t) = 1(t) \xrightarrow{\mathcal{L}} U(s) = \frac{1}{s}$$

$$Y(s) = K \frac{1}{s(1 + Ts)} \xrightarrow{\mathcal{L}^{-1}}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \frac{K}{T} \cdot \mathcal{L}^{-1}\left\{\frac{1}{s} \cdot \frac{1}{s + \frac{1}{T}}\right\} = \frac{K}{T} \cdot (1(t) * e^{-\frac{t}{T}}) = K(1 - e^{-t/T})$$

for the values $K = 5$, $T = 4$ the solution is depicted in the above picture.

4 Solve the state space equation

- only excitation ($x_0 = 0, u(t) \neq 0$) \rightarrow : Laplace transform
- only initial value ($x_0 \neq 0, u(t) = 0$) $\rightarrow e^{At}x_0$, state trajectories
- both excitation and initial values

Example 8. SSM solution – unit step input

$$\begin{aligned}
 A &= \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} & B &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} & C &= (0 \ 1) & x_0 &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} & u(t) &= 1(t) \\
 \dot{x} &= Ax + Bu \\
 y &= Cx
 \end{aligned} \tag{35}$$

Applying Laplace transformation:

$$\begin{aligned}
 sX(s) &= AX(s) + BU(s) \rightarrow sX(s) - AX(s) = BU(s) \\
 (sI - A)X(s) &= BU(s) \rightarrow X(s) = (sI - A)^{-1}BU(s) \\
 Y(s) &= C(sI - A)^{-1}BU(s) \\
 H(s) &= Y(s)/U(s) = C(sI - A)^{-1}B = \frac{s}{s^2 - 3s - 4} = \frac{s}{(s+1)(s-4)} \\
 Y(s) &= H(s)U(s) = \frac{s}{(s+1)(s-4)} \cdot \frac{1}{s} = \frac{1}{(s+1)(s-4)} = \frac{0.2}{s-4} - \frac{0.2}{s+1} \\
 y(t) &= 0.2e^{4t} - 0.2e^{-t}
 \end{aligned}$$

Example 9. SSM solution – autonomous system

$$A = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad C = (0 \ 1) \quad x_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad u(t) = 0$$

Applying Laplace transformation:

$$sX(s) - x_0 = AX(s) \rightarrow X(s) = (sI - A)^{-1}x_0 \rightarrow x(t) = \mathfrak{L}^{-1}\{(sI - A)^{-1}\}x_0 = e^{At}x_0$$

$$(sI - A)^{-1} = \frac{1}{(s+1)(s-4)} \cdot \begin{pmatrix} s-1 & 3 \\ 2 & s-2 \end{pmatrix}$$

$$\text{Output: } y(t) = Cx(t) = C \cdot \mathfrak{L}^{-1}\{(sI - A)^{-1}\} \cdot x_0 = \mathfrak{L}^{-1}\{C(sI - A)^{-1}x_0\} = \mathfrak{L}^{-1}\left\{\frac{s-2}{(s+1)(s-4)}\right\} = 0.6e^{-t} + 0.4e^{4t}$$

Example 10. SSM solution – unit step velocity

$$A = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad C = (0 \quad 1) \quad x_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad u(t) = t$$

Applying Laplace transformation:

$$sX(s) - x_0 = X(s) + BU(s) \quad \rightarrow \quad X(s) = (sI - A)^{-1}(x_0 + BU(s)) \quad \rightarrow$$

$$x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^t e^{A(t-\tau)}Bu(\tau)d\tau$$

If $t_0 = 0$, then $e^{A(t-t_0)} = e^{At}$

$$e^{At} = \mathfrak{L}^{-1} \left\{ \begin{pmatrix} \frac{s-1}{s^2-3s-4} & \frac{3}{s^2-3s-4} \\ \frac{2}{s^2-3s-4} & \frac{s-2}{s^2-3s-4} \end{pmatrix} \right\} = \mathfrak{L}^{-1} \left\{ \begin{pmatrix} \frac{0.6}{s-4} + \frac{0.4}{s+1} & \frac{0.6}{s-4} - \frac{0.6}{s+1} \\ \frac{0.4}{s-4} - \frac{0.4}{s+1} & \frac{0.4}{s-4} + \frac{0.6}{s+1} \end{pmatrix} \right\}$$

$$e^{At} = \begin{pmatrix} 0.6e^{4t} + 0.4e^{-t} & 0.6e^{4t} - 0.6e^{-t} \\ 0.4e^{4t} - 0.4e^{-t} & 0.4e^{4t} + 0.6e^{-t} \end{pmatrix}$$

$$e^{A(t-\tau)} = \begin{pmatrix} 0.6e^{4(t-\tau)} + 0.4e^{-(t-\tau)} & 0.6e^{4(t-\tau)} - 0.6e^{-(t-\tau)} \\ 0.4e^{4(t-\tau)} - 0.4e^{-(t-\tau)} & 0.4e^{4(t-\tau)} + 0.6e^{-(t-\tau)} \end{pmatrix}$$

$$e^{A(t-\tau)}B = \begin{pmatrix} 1.2e^{4(t-\tau)} - 0.2e^{-(t-\tau)} \\ 0.8e^{4(t-\tau)} + 0.2e^{-(t-\tau)} \end{pmatrix} \rightarrow e^{A(t-\tau)}Bu(\tau) = \begin{pmatrix} 1.2e^{4(t-\tau)}\tau - 0.2e^{-(t-\tau)}\tau \\ 0.8e^{4(t-\tau)}\tau + 0.2e^{-(t-\tau)}\tau \end{pmatrix}$$

Elementwise integral:

$$\int_0^t c_1 e^{c_2(t-\tau)}\tau d\tau = c_1 e^{c_2 t} \int_0^t e^{-c_2\tau}\tau d\tau = \frac{c_1}{c_2^2} (e^{c_2 t} - c_2 t - 1) \quad (\text{Partial integration})$$

$$\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau = \begin{pmatrix} 0.075e^{4t} - 0.2e^{-t} - 0.5t + 0.125 \\ 0.05e^{4t} + 0.2e^{-t} - 0.25 \end{pmatrix}$$

$e^{At}x_0$ the same value as in the case of 2. example

$$x(t) = \begin{pmatrix} 0.675e^{4t} - 0.8e^{-t} - 0.5t + 0.125 \\ 0.45e^{4t} + 0.8e^{-t} - 0.25 \end{pmatrix}$$

$$y(t) = Cx(t) = 0.45e^{4t} + 0.8e^{-t} - 0.25$$

A 2. we can see the solution of the three example in order.

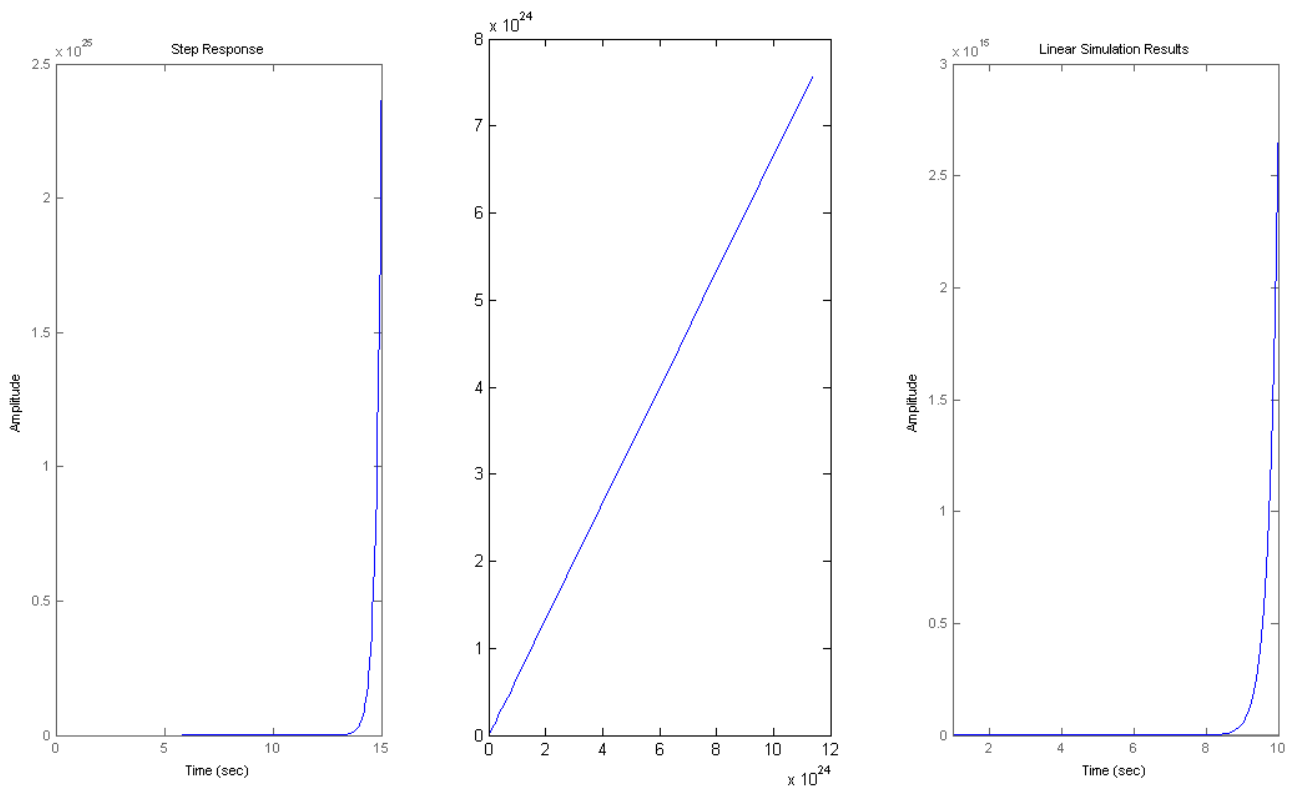


Figure 2