Partial fractional decomposition (residuum approach)

$$H(s) = \frac{5s^2+3s+1}{s^3+6s^2+11s+6} = \frac{5s^2+3s+1}{(s+1)(s+2)(s+3)} \tag{1}$$

$$\frac{5s^2+3s+1}{(s+1)(s+2)(s+3)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s+3}\Bigg| \cdot (s+1) \tag{2}$$

$$\frac{5s^2+3s+1}{(s+2)(s+3)} = A + (s+1)\left(\frac{B}{s+2} + \frac{C}{s+3}\right) \leftarrow \boxed{s=-1} \tag{3}$$

$$\frac{5-3+1}{2} = \frac{3}{2} = A \tag{4}$$

...

$$H(s) = \frac{3}{2(s+1)} - \frac{15}{s+2} + \frac{37}{2(s+3)} \tag{5}$$

Partial fractional decomposition ("brute force" approach)

$$Y(s) = \frac{2s^2 + 10s}{(s^2 - 2s + 5)(s + 1)} = \frac{As + B}{s^2 - 2s + 5} + \frac{C}{s + 1} \tag{6}$$

we multiply by the denominators

$$2s^2 + 10s = As^2 + (A + B)s + B + Cs^2 - 2Cs + 5C \tag{7}$$

$$2s^2 + 10s = (A + C)s^2 + (A + B - 2C)s + (B + 5C) \tag{8}$$

$$\begin{cases} A + C = 2 \\ A + B - 2C = 10 \\ B + 5C = 0 \end{cases} \Rightarrow \begin{cases} A = 2 - C \\ 2 - C - 5C - 2C = 10 \\ B = -5C \end{cases} \Rightarrow \begin{cases} A = 3 \\ B = 5 \\ C = -1 \end{cases} \tag{9}$$

(Almost) finally:

$$Y(s) = \frac{3s + 5}{s^2 - 2s + 5} - \frac{1}{s + 1} \tag{10}$$

$$\frac{3s + 5}{s^2 - 2s + 1 + 4} = \frac{3s + 5}{(s - 1)^2 + 2^2} = 3 \cdot \frac{s - 1}{(s - 1)^2 + 2^2} + 4 \cdot \frac{2}{(s - 1)^2 + 2^2} \tag{11}$$

Finally: $\boxed{y(t) = 3e^t \cos(2t) + 4e^t \sin(2t) - e^{-t}}$.

Matrix exponential using eigenvalue decomposition:

$$A = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} \Rightarrow e^A = ? \tag{12}$$

$$A = SDS^{-1} \Rightarrow A^2 = SDS^{-1}SDS^{-1} = SD^2S^{-1} \Rightarrow A^n = SD^nS^{-1} \tag{13}$$

$$e^A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n = S\left(\sum_{n=0}^{\infty} \frac{1}{n!} D^n\right) S^{-1} = Se^D S^{-1} \tag{14}$$

$$D^n = \begin{pmatrix} \lambda_1^n & & & \\ & \lambda_2^n & & \\ & & \ddots & \\ & & & \lambda_m^n \end{pmatrix} \Rightarrow e^D = \begin{pmatrix} e^{\lambda_1} & & & \\ & e^{\lambda_2} & & \\ & & \ddots & \\ & & & e^{\lambda_m} \end{pmatrix} \tag{15}$$

If we have an additional time (t) multiplier:

$$e^{At} = Se^{Dt}S^{-1}, \text{ where } e^{Dt} = \begin{pmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_m t} \end{pmatrix} \tag{16}$$

Matrix exponential using (direct and inverse) Laplace transformation:

$$\boxed{e^{At} = \mathfrak{L}^{-1}\{(sI - A)^{-1}\}} \tag{17}$$

e.g.

$$A = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} \Rightarrow e^{At} =? \tag{18}$$

Then,

$$\det(sI - A) = \begin{vmatrix} s-2 & -3 \\ -2 & s-1 \end{vmatrix} = (s-2)(s-1) - 6 = s^2 - 3s - 4 = (s-4)(s+1) \tag{19}$$

$$(sI - A)^{-1} = \frac{1}{(s-4)(s+1)} \begin{pmatrix} s-1 & 3 \\ 2 & s-2 \end{pmatrix} \tag{20}$$

$$s^3Y(s) - 2 + 7s^2Y(s) + 14sY(s) + 8Y(s) = 0 \tag{21}$$

$$(s^3 + 7s^2 + 14s + 8)Y(s) = 2 \tag{22}$$

$$Y(s) = \frac{2}{s^3 + 7s^2 + 14s + 8} \tag{23}$$

$$s^3 + 7s^2 + 14s + 8 = (s+1)(s^2 + 6s + 8) = (s+1)(s+2)(s+4) \tag{24}$$

$$\frac{2}{s^3 + 7s^2 + 14s + 8} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s+4}\Bigg| \cdot (s+1) \tag{25}$$

$$\Rightarrow A = \frac{2}{(s+2)(s+4)}\Big|_{s=-1} \tag{26}$$

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \tag{27}$$

$$\dot{x} = Ax \Rightarrow sX(s) - x(0) = AX(s) \Rightarrow sX(s) - AX(s) = x(0) \tag{28}$$

$$(sI - A)X(s) = x(0) \tag{29}$$

$$X(s) = (sI - A)^{-1}x(0) \tag{30}$$

$$x(t) = \mathfrak{L}^{-1}\left\{(sI - A)^{-1}x(0)\right\} = \mathfrak{L}^{-1}\left\{(sI - A)^{-1}\right\}x(0) \tag{31}$$

$$\boxed{x(t) = e^{At}x(0)} \tag{32}$$