Computer controlled systems

Basic linear algebra operations in Matlab

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1 Problems

1. Given an orthonormal basis of the image space of matrix
$$A = \begin{pmatrix} -2 & -1 & 7 \\ -3 & -5 & 14 \\ 1 & 2 & -5 \end{pmatrix}.$$
2. Give on orthonormal basis of the kernel (or null) space of matrix A.
3. Give on orthonormal basis of $\mathrm{Im}(A)^\perp$, the orthogonal complement of the image space of matrix A.
4. Give on orthonormal basis of the kernel space of matrix A^T.
5. Given on orthonormal basis of the image space of matrix A^T.
6. Compute the singular value decomposition of matrix A.
7. Compute the rank of diagonal matrix S and matrix A.
8. Compute the intersection of $V = \mathrm{Im}(A)$ and $W = \mathrm{Im}(A^T)$. Note that $V \cap W = \left(V^\perp \cup W^\perp\right)^\perp$
9. Give a matrix for which $\mathrm{Im}(A) = \mathrm{Ker}(A)$.
10. Compute the partial fraction decomposition of $H_1(s) = \frac{s}{s^2+3s+2}$ with `residue` and `partfrac`.
11. Compute the partial fraction decomposition of $H_2(s) = \frac{s^3+3s+1}{(s^2+4)(s-2)(s+3)(s^2+2s+2)}$ with ...

2 Fundamental theorem of linear algebra

Theorem 1. The fundamental theorem of linear algebra

Let $\mathcal{A} : \mathbb{R}^n \to \mathbb{R}^m$, $\mathcal{A}(x) = Ax$, where $A \in \mathbb{R}^{m\times n}$. Then the followings are true

$$\mathrm{Im}(A) = \mathrm{Ker}\left(A^T\right)^\perp \subset \mathbb{R}^m \tag{1a}$$
$$\mathrm{Im}\left(A^T\right) = \mathrm{Ker}(A)^\perp \subset \mathbb{R}^n \tag{1b}$$

Furthermore

$$\mathrm{Im}(A) \otimes \mathrm{Ker}\left(A^T\right) = \mathbb{R}^m \tag{2a}$$
$$\mathrm{Im}\left(A^T\right) \otimes \mathrm{Ker}(A) = \mathbb{R}^n \tag{2b}$$

Remark. If $r = \mathrm{rank}(A)$, than

$$\dim \mathrm{Im}(A) = r, \qquad \dim \mathrm{Ker}\left(A^T\right) = m - r \tag{3a}$$
$$\dim \mathrm{Im}\left(A^T\right) = r, \qquad \dim \mathrm{Ker}(A) = n - r \tag{3b}$$

Proof. Proof of (1a) as presented in [1]. Let

$$A = \begin{pmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \dots & \boldsymbol{a}_n \end{pmatrix} \;\Rightarrow\; A^T = \begin{pmatrix} \boldsymbol{a}_1^T \\ \boldsymbol{a}_2^T \\ \vdots \\ \boldsymbol{a}_n^T \end{pmatrix} \tag{4a}$$

$$\boldsymbol{x} \in \mathrm{Ker}\big(A^T\big) \;\Rightarrow\; A^T\boldsymbol{x} = \begin{pmatrix} \boldsymbol{a}_1^T\boldsymbol{x} \\ \boldsymbol{a}_2^T\boldsymbol{x} \\ \vdots \\ \boldsymbol{a}_n^T\boldsymbol{x} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{4b}$$

$$\boldsymbol{y} \in \mathrm{Im}(A) \;\Rightarrow\; \exists \alpha_i \in \mathbb{R} \text{ such that } \boldsymbol{y} = \sum_{i=1}^{n} \alpha_i \boldsymbol{a}_i \tag{4c}$$

Note that $\boldsymbol{x}$ and $\boldsymbol{y}$ are arbitrary vector elements of $\mathrm{Ker}\big(A^T\big)$ and $\mathrm{Im}(A)$, respectively. Then we compute the dot product of $\boldsymbol{x}$ and $\boldsymbol{y}$:

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{y}^T\boldsymbol{x} = \sum_{i=1}^{n} \alpha_i \boldsymbol{a}_i^T \boldsymbol{x} = 0, \tag{5}$$

since $\boldsymbol{a}_i^T\boldsymbol{x} = 0$, $\forall i = \overline{1,n}$. Consequently, $\boldsymbol{x} \perp \boldsymbol{y}$ for all possible $x \in \mathrm{Ker}\big(A^T\big)$ and $y \in \mathrm{Im}(A)$, which means that the two subspaces are the ***orthogonal complement*** for each other:

$$\begin{aligned} \mathrm{Im}(A) &= \mathrm{Ker}\big(A^T\big)^{\perp} \\ \mathrm{Im}(A) \cap \mathrm{Ker}\big(A^T\big) &= \{0\} \end{aligned} \tag{6}$$

Following this idea, we can conclude that the subspaces are linearly independent, therefore,

$$\dim\Big(\mathrm{Im}(A) \otimes \mathrm{Ker}\big(A^T\big)\Big) = r + (m - r) = m. \tag{7}$$

This can only happend if ***direct product*** of the two spaces is $\mathbb{R}^m$, which completes the proof for (2a). □

Proposition 2. (Self-adjoint operator) Let $\mathcal{A} : \mathbb{R}^n \to \mathbb{R}^n$, $\mathcal{A}(x) = Ax$, where $A \in \mathbb{R}^{n\times n}$ is a symmetric matrix: $A = A^T$. Than, as a consequence of Theorem 1, we have that

$$\mathrm{Im}(A) = \mathrm{Ker}(A)^{\perp} \text{ and } \mathrm{Im}(A) \otimes \mathrm{Ker}(A) = \mathbb{R}^n.$$

For more, see [2, Eq. (10.3)].

Proposition 3. Singular value decomposition (SVD)

If we make the SVD for matrix $A \in \mathbb{R}^{m\times n}$

$$A = U\Sigma V^T, \tag{8}$$

where

$$U \in \mathbb{R}^{m\times m} \text{ is unitary: } U^*U = I_m \tag{9a}$$

$$V \in \mathbb{R}^{n\times n} \text{ is unitary: } V^*V = I_n \tag{9b}$$

$$\Sigma \in \mathbb{R}^{m\times n} \text{ eigenvalues in the diagonal.} \tag{9c}$$

After this decomposition, the basis of the four subspaces (2) can be obtained as presented below.

$$\begin{aligned} &\mathrm{Im}(A): && \text{the first } r \text{ colums of } U \\ &\mathrm{Ker}\big(A^T\big): && \text{the last } m-r \text{ columns of } U \\ &\mathrm{Im}\big(A^T\big): && \text{the first } r \text{ columns of } V \\ &\mathrm{Ker}(A): && \text{the last } n-r \text{ columns of } V \end{aligned}$$

In short

$$\text{``}\, A = \big[\mathrm{Im}(A) \quad \mathrm{Ker}\big(A^T\big)\big]\, \Sigma\, \big[\mathrm{Im}\big(A^T\big) \quad \mathrm{Ker}(A)\big]^T \,\text{''} \tag{10}$$

References

[1] Alexey Grigorev. The Fundamental Theorem of Linear Algebra. Technische Universität Berlin.

[2] Lantos Béla. *Irányítási rendszerek elmélete és tervezése I.* Akadémiai Kiadó Budapest, 2001.