Computer controlled systems

Lecture 7, March 31, 2017

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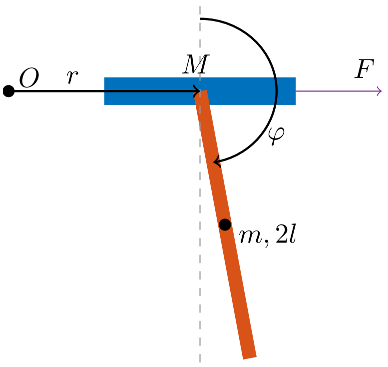
Inverted pendulum model

We consider a simple pendulum mounted an a chart that can move horizontally:



M	is the mass of the chart
m	is the mass of the pendulum
$2l$	is the length of the pendulum
l	is the distance of the pivot point from the pendulum's center of mass
F	is an external force (input) acting on the chart
b	is the damping factor
r	is the (horizontal) position of the chart
$\dot{r} = v$	is the (horizontal) velocity of the chart
ϕ	is the angle of the chart (clockwise direction)
$\dot{\phi} = \omega$	is the angular velocity of the chart (clockwise direction)
$\phi = 0$	**unstable equilibrium point**: if the pendulum's center of mass is exactly **above** its pivot point (is vertical and pointing towards the sky)
$\phi = \pi$	**stable equilibrium point**: if the pendulum's center of mass is exactly **below** its pivot point

This system has a nonlinear equation, which can be linearized in a certain operating point[1] (see Appendix). The state vector of the system is the following: $x = \begin{pmatrix} r & v & \phi & \omega \end{pmatrix}^T$, furthermore, the external force F constitutes the input of the system (u). The nonlinear model of the system is: $\dot{x} = f(x) + g(x)u$, where

$$f(x) = \begin{pmatrix} v \\ \frac{1}{q}\Big(4ml\sin(\phi)\omega^2 - 1.5mg\sin(2\phi) - 4bv\Big) \\ \omega \\ \frac{3}{lq}\Big(-\frac{ml}{2}\sin(2\phi)\omega^2 + (M+m)g\sin(\phi) + b\cos(\phi)v\Big) \end{pmatrix}, \quad g(x) = \frac{1}{lq}\begin{pmatrix} 0 \\ 4l \\ 0 \\ -3\cos(\phi) \end{pmatrix} \tag{1}$$

where $q = 4(M+m) - 3m\cos(\phi)^2$. For the full derivation see Appendix. For each exercise, you can use your own parameter configuration. Some examples are listed below.

(A) no friction	**(B)** with friction	**(C)** with friction + heavy rod
$M =$ 0.5 [kg]	$M =$ 0.5 [kg]	$M =$ 0.5 [kg]
$m =$ 0.2 [kg]	$m =$ 0.2 [kg]	$m =$ 10 [kg]
$l =$ 1 [m]	$l =$ 1 [m]	$l =$ 1 [m]
$g =$ 9.8 [m/s^2]	$g =$ 9.8 [m/s^2]	$g =$ 9.8 [m/s^2]
$b =$ 0 [kg/s]	$b =$ 10 [kg/s]	$b =$ 10 [kg/s]

[1] munkapont

Linearized model around the *unstable* equilibrium point ($\phi = 0$)

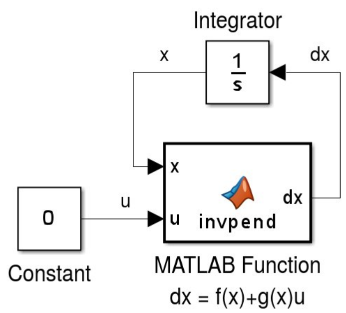
Linearized state space model around the unstable operating point $x^* = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}^T$ is:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{4b}{4M+m} & -\frac{3mg}{4M+m} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{3b}{l(4M+m)} & \frac{3(M+m)g}{l(4M+m)} & 0 \end{pmatrix}, \quad B = \frac{1}{l(4M+m)} \begin{pmatrix} 0 \\ 4l \\ 0 \\ -3 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{2}$$

Exercises

1. Simulate the motion of the inverted pendulum in Simulink, use the original **nonlinear** model of the system.
   Instructions.

   - Using the Simulink's "MATLAB function", you can implement the equation $\dot{x} = f(x) + g(x)u$ as a Matlab function $\texttt{dx} = \texttt{invpend}(\texttt{x}, \texttt{u})$ with two input arguments (the state variables $x \in \mathbb{R}^4$ and input $u \in \mathbb{R}$) and a single output argument ($\dot{x} \in \mathbb{R}^4$ the time derivative of x)
   - The time derivative of $\dot{x}$ is fed back through an integrator (see figure below).
   - In order to plot the result, use the "Scope" block diagram.
   - If you want to export the numerical values to the Matlab's global workspace use "To Workspace" block.
   - The initial value of the system can be given as the initial value of the integrator: open the "Block Parameters" dialog of the integrator.



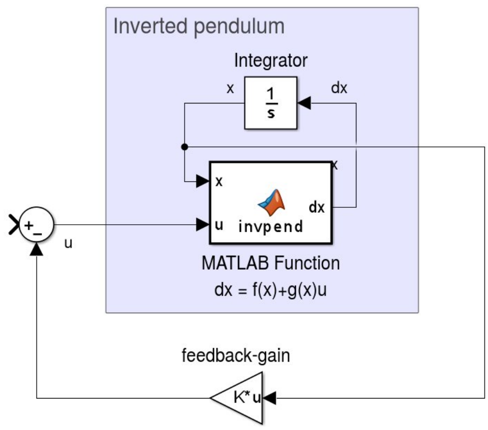
2. Design a state feedback gain in Matlab for the (**linearized**) system, which

   (a) translates the poles into $\{-1, -2, -3, -4\}$ (or into arbitrary stable poles).

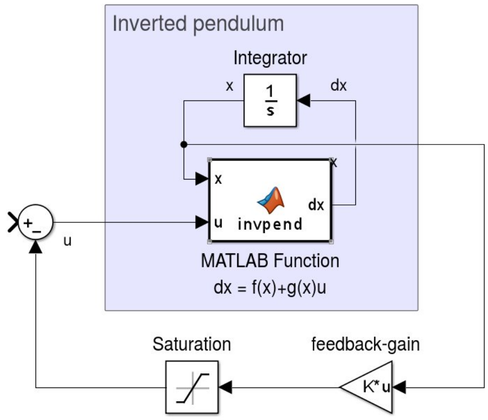
   (b) minimizes the functional $J(x, u) = \int_0^\infty x^T Q x + u^T R u \, \mathrm{d}t$, where $Q = I_4$ and $R = 1$ (LQR design).

3. Apply the state feedback gain on the **nonlinear** model, and simulate it in Simulink.
   Instructions.

   - Use the "Gain" block of Simulink, open its "Block Parameters" dialog, and type there the value of the obtained K.
   - Be aware that the multiplication rule is set to be "Matrix(K*u)" (i.e. matrix by matrix multiplication).



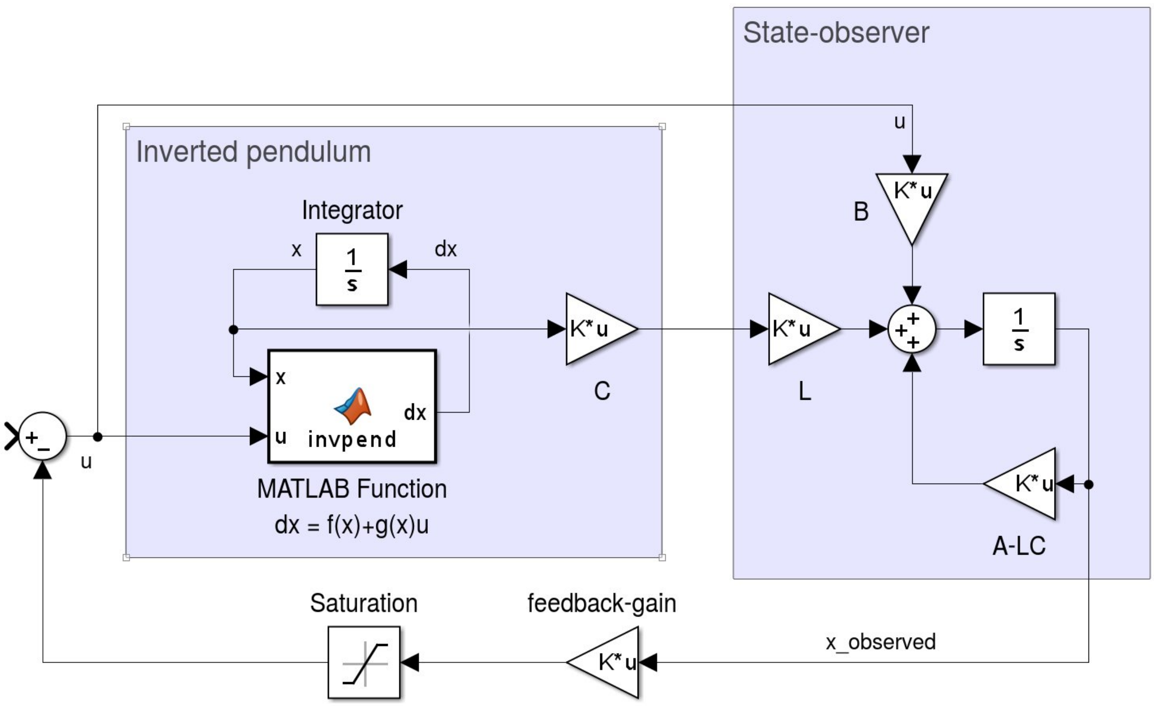
4. In practical applications the actuator has a finite power to act on the system, so it cannot execute arbitrarily large input values. Simulate this saturation effect in Simulink using the "Saturation" block.



5. Design a stable state observer in Matlab for the (**linearized**) system.

6. Simulate the nonlinear system with the existing static feedback of the **observed** state vector $\hat{x}$.

   - Optionally, you can add Gaussian noise to the input (actuator noise) or to the output (sensor noise). Use the "Gaussian Noise Generator" block.



Appendix

I. Linearize a nonlinear model around an equilibrium point

We have a nonlinear system in the following form:

$$\dot{x} = F(x,u) = f(x) + g(x)u \tag{3}$$

Let $x^* \in \mathbb{R}^n$ be an equilibrium point of the nonlinear system, which means that $F(x^*,0) = f(x^*) = 0$. We assume that the system operates around this equilibrium point, and by default there is no input given to the system. Therefore, we say that the system's operating point[2] is $(x^*, u^* = 0)$.

The Jacobian matrix of $F(x,u)$ is

$$\mathrm{D}[F(x,u)] = \left(\tfrac{\partial F(x,u)}{\partial x} \;\middle|\; \tfrac{\partial F(x,u)}{\partial u}\right) = \left(\tfrac{\partial f(x)}{\partial x} + \tfrac{\partial g(x)}{\partial x}u \;\middle|\; g(x)\right) \tag{4}$$

The value of the Jacobian matrix in this operating point is

$$\mathrm{D}[F(x^*,0)] = \left(\tfrac{\partial f(x^*)}{\partial x} \;\middle|\; g(x^*)\right) \tag{5}$$

Now we estimate $F(x,u)$ by its first order Taylor polynomial around the operating point:

$$\begin{aligned} F(x,u) &\simeq \underbrace{F(x^*,0)}_{0} + \mathrm{D}[F(x^*,0)] \begin{pmatrix} x - x^* \\ u - 0 \end{pmatrix} \\ F(x,u) &\simeq \frac{\partial f(x^*)}{\partial x}(x - x^*) + g(x^*)u \end{aligned} \tag{6}$$

Hence, the linear model is

$$\dot{x} = A(x - x^*) + Bu, \quad \text{where} \quad \begin{aligned} A &:= \frac{\partial f(x^*)}{\partial x} \\ B &:= g(x^*) \end{aligned} \tag{7}$$

There's only one more thing left, we need to center the system. We introduce the centered state vector $\bar{x} := x - x^*$. Therefore, the time derivative of the transformed state vector will be:

$$\dot{\bar{x}} = \dot{x} = A(x - x^*) + Bu = A\bar{x} + Bu \tag{8}$$

Finally, we obtained the centered linearized model:

$$\dot{\bar{x}} = A\bar{x} + Bu, \quad \text{where} \quad \begin{aligned} A &:= \frac{\partial f(x^*)}{\partial x} \\ B &:= g(x^*) \end{aligned} \tag{9}$$

II. Derivation of the inverted pendulum's equation

The equation of the inverted pendulum is the following:

$$\begin{aligned} (M+m)\ddot{x} + ml\ddot{\phi}\cos(\phi) - ml\dot{\phi}^2\sin(\phi) &= F \\ ml\ddot{x}\cos(\phi) + \frac{4}{3}ml^2\ddot{\phi} - mgl\sin(\phi) &= 0 \end{aligned} \tag{10}$$

The nonlinear state space equation of the inverted pendulum:

$$\begin{cases} \dot{x} = v \\ \dot{v} = \dfrac{1}{q}\big(4ml\sin(\phi)\omega^2 - 1.5mg\sin(2\phi) - 4bv\big) + \dfrac{4}{q}F \\ \dot{\phi} = \omega \\ \dot{\omega} = \dfrac{3}{lq}\left(-\dfrac{ml}{2}\sin(2\phi)\omega^2 + (M+m)g\sin(\phi) + b\cos(\phi)v\right) - \dfrac{3\cos(\phi)}{lq}F \end{cases} \tag{11}$$

[2] munkapont

where $q = 4(M+m) - 3m\cos(\phi)^2$. Let the state vector be $x = \begin{pmatrix} x & v & \phi & \omega \end{pmatrix}^T$.

$$f(x) = \begin{pmatrix} v \\ \frac{1}{q}\Big(4ml\sin(\phi)\omega^2 - 1.5mg\sin(2\phi) - 4bv\Big) \\ \omega \\ \frac{3}{lq}\Big(-\frac{ml}{2}\sin(2\phi)\omega^2 + (M+m)g\sin(\phi) + b\cos(\phi)v\Big) \end{pmatrix} \quad , \quad g(x) = \frac{1}{lq}\begin{pmatrix} 0 \\ 4l \\ 0 \\ -3\cos(\phi) \end{pmatrix} \tag{12}$$

Linearized model around the stable operating point $x^* = \begin{pmatrix} 0 & 0 & \pi & 0 \end{pmatrix}^T$:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{4b}{4M+m} & -\frac{3mg}{4M+m} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{3b}{l(4M+m)} & -\frac{3(M+m)g}{l(4M+m)} & 0 \end{pmatrix} \quad , \quad B = \frac{1}{l(4M+m)}\begin{pmatrix} 0 \\ 4l \\ 0 \\ 3 \end{pmatrix} \quad , \quad C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{13}$$

Linearized state space model around the unstable operating point $x^* = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}^T$ is:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{4b}{4M+m} & -\frac{3mg}{4M+m} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{3b}{l(4M+m)} & \frac{3(M+m)g}{l(4M+m)} & 0 \end{pmatrix} \quad , \quad B = \frac{1}{l(4M+m)}\begin{pmatrix} 0 \\ 4l \\ 0 \\ -3 \end{pmatrix} \quad , \quad C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{14}$$