Computer controlled systems

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Pole-placement controller

Pole-placement controller based on Bass-Gura formula

$$K = (\underline{\alpha} - \underline{a})T_l^{-1}\mathcal{C}^{-1}$$

where $\underline{\alpha}$ is the expected (prescribed) characteristic polynomial of the closed-loop system, $\underline{a}$ is the characteristic polynomial of the original (uncontrolled) system, $\mathcal{C}$ is the controllability matrix, finally T_l is the following Toeplitz matrix:

$$T_l = \begin{pmatrix} 1 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & 1 & a_1 & \cdots & a_{n-2} \\ 0 & 0 & 1 & \cdots & a_{n-3} \\ . & . & . & \cdots & . \end{pmatrix}$$

Example 1. Design a pole-placement controller for the following CT LTI SISO system:

$$A = \begin{pmatrix} 2 & -2 \\ 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 1 \end{pmatrix}$$

Solution.

$$\begin{aligned} a(s) &= s^2 - 3s + 2 \\ a_1 &= -3 \\ a_2 &= 2 \end{aligned}$$

The prescribed characteristic polynomial ($\phi_c(s)$):

$$\begin{aligned} \alpha(s) &= s^2 + 3s + 2 \\ \alpha_1 &= 3 \\ \alpha_2 &= 2 \end{aligned}$$

A Toeplitz matrix and the controllability matrix in this case are

$$T_l = \begin{pmatrix} 1 & a_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix} \qquad \mathcal{C} = \begin{pmatrix} 1 & -2 \\ 2 & 2 \end{pmatrix}$$

$$T_l^{-1} = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \qquad \mathcal{C}^{-1} = \frac{1}{6}\begin{pmatrix} 2 & 2 \\ -2 & 1 \end{pmatrix}$$

Than the static state feedback will be the following:

$$K = \begin{pmatrix} 3-(-3) & 2-2 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \frac{1}{6} \begin{pmatrix} 2 & 2 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -4 & 5 \end{pmatrix}$$

Ackermann formula

$$K = [0\,0\,\cdots\,0\,1]\mathcal{C}_n^{-1}\phi_c(A)$$

where $\phi_c(s)$ is the prescribed characteristic polynomial of the closed-loop (controlled) system. In the previous example, it was denoted by $\alpha(s) = \phi_c(s)$.

Example 2. Design a pole-placement controller for the following CT LTI SISO system:

$$A = \begin{pmatrix} 2 & -2 \\ 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 1 \end{pmatrix}$$

Solution.

$$\mathcal{C}_2 = \begin{pmatrix} B & AB \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 2 & 2 \end{pmatrix} \to \mathcal{C}_2^{-1} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{6} \end{pmatrix}$$

Legyen $\lambda_1 = -1$ és $\lambda_2 = -2$.

$$\phi_c = (s - \lambda_1)(s - \lambda_2) = s^2 + 3s + 2$$

$$\phi_c(A) = A^2 + 3A + 2I = \begin{pmatrix} 12 & -12 \\ 0 & 6 \end{pmatrix}$$

$$K = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{6} \end{pmatrix} \begin{pmatrix} 12 & -12 \\ 0 & 6 \end{pmatrix} = \begin{pmatrix} -4 & 5 \end{pmatrix}$$

Check

$$A - BK = \begin{pmatrix} 2 & -2 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} -4 & 5 \end{pmatrix} = \begin{pmatrix} 6 & -7 \\ 8 & -9 \end{pmatrix}$$

$$\det(\lambda I - (A - BK)) = \lambda^2 + 3\lambda + 2$$

Namely, the poles of the obtained closed-loop system are indeed the prescribed values.

Example 3. Design a pole-placement controller for the following CT LTI SISO system:

$$A = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 1 \end{pmatrix}$$

Solution.

$$\mathcal{C}_2 = \begin{pmatrix} B & AB \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \to \mathcal{C}_2^{-1} = \begin{pmatrix} 1 & -\frac{2}{3} \\ 0 & \frac{1}{3} \end{pmatrix}$$

Let $\lambda_1 = -1$ and $\lambda_2 = -2$.

$$\phi_c = (s + \lambda_1)(s + \lambda_2) = s^2 + 3s + 2$$

$$\phi_c(A) = A^2 + 3A + 2I = \begin{pmatrix} 9 & -3 \\ 9 & -3 \end{pmatrix}$$

$$K = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{2}{3} \\ 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 9 & -3 \\ 9 & -3 \end{pmatrix} = \begin{pmatrix} 3 & -1 \end{pmatrix}$$

Check:

$$A - BK = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 3 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 3 & -2 \end{pmatrix}$$

$$\det(\lambda I - (A - BK)) = \lambda^2 + 3\lambda + 2$$

Indeed, the poles of the closed loop system are the prescribed values.

Example 4. Given the following CT LTI SISO systems

1. $\begin{cases} \dot{x} = \begin{pmatrix} 2 & 0 \\ 9 & -3 \end{pmatrix} x + \begin{pmatrix} 0 \\ 3 \end{pmatrix} u \\ y = \begin{pmatrix} 1 & 1 \end{pmatrix} x \end{cases}$

2. $\begin{cases} \dot{x} = \begin{pmatrix} 2 & 0 \\ 9 & -3 \end{pmatrix} x + \begin{pmatrix} 2 \\ 3 \end{pmatrix} u \\ y = \begin{pmatrix} 1 & 1 \end{pmatrix} x \end{cases}$

Design a state feedback controller (if it is possible), that stabilizes the system!

Example 5. Given the following CT LTI SISO system $H(s) = \frac{2s-4}{s^2+s-6}$.

1. Is the system asymptotically stable?
2. If it is possible, design a controller, that shifts the system's poles to $p_1 = -3$ and $p_2 = -5$! Hint: controllability normal form.

Linear state observer design

Goal: computation of the values of the non-measured state variables of the system using the observed output.

The dynamical system

$$\frac{\mathrm{d}\hat{x}}{\mathrm{d}t} = F\hat{x} + Ly + Hu$$

is called a full order state observer, if the error dynamics $e = x - \hat{x}$ tends to zero, i.e. $\lim_{t\to\infty} e = 0$
In case of an LTI system:

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

$$\dot{e} = \dot{x} - \dot{\hat{x}} = Ax + Bu - F\hat{x} - Ly - Hu + Fx - Fx =$$
$$= Ax + Bu - F\hat{x} - LCx - Hu + Fx - Fx =$$
$$= (A - LC - F)x + (B - H)u + F(x - \hat{x}) = (A - LC - F)x + (B - H)u + F(e)$$

Let $F = A - LC$ and $H = B$
Than $\dot{e} = Fe$
We require that the system be asymptotically stable, namely the real part of the roots of the characteristic polynomial $\det(sI - (A - LC))$ be negative.

$$\det(sI - (A - LC)) = \det\big(sI - (A^T - C^T L^T)\big)$$

We can observe that the state observer design can be traced back to a pole placement problem of (A', B'), where $A' = A^T$, $B' = C^T$, and the result (K) of the pole placement should be interpreted as $L = K^T$.

Example 6. Design a state observer for the following CT LTI SISO system

$$A = \begin{pmatrix} -3 & 1 \\ 2 & -1 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad C = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

Solution.
Let the characteristic polynomial of the closed-loop system: $\phi_o(s) = (s+3)(s+3)$
In order to use the Ackermann, formula we should substitute $A' = A^T$ into $\phi_o(s)$:

$$\phi_o(A') = \begin{pmatrix} 2 & 4 \\ 2 & 6 \end{pmatrix}$$

If $B' = C^T$, the obtained controllability matrix for (A', B') (which is actually the transpose of the observability matrix of (A, C)) is:

$$\mathcal{C}_2' = \begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix}$$

Its inverse will be:

$$(\mathcal{C}_2')^{-1} = \begin{pmatrix} 1/2 & 1 \\ 1/2 & 0 \end{pmatrix}$$

Finally, we compute the feedback gain K:

$$K = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 1/2 & 1 \\ 1/2 & 0 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 2 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 2 \end{pmatrix}$$

From this:

$$L = K^T = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad F = A - LC = \begin{pmatrix} -3 & 0 \\ 2 & -3 \end{pmatrix} \quad H = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Example 7. Design a state observer for the following CT LTI SISO system

$$A = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

Example 8. Design a state observer AND a stabilizer state feedback controller for the following CT LTI SISO system.

$$A = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

Separation principle: the observer gain L and the feedback gain K can be designed separately.

Optimal state feedback controller - LQR controller design

We want to minimize the following functional:

$$J(x,u) = \frac{1}{2}\int_0^T x^TQx + u^TRu\ dt$$

where Q and R are positive definite symmetric matrices. In case of LTI systems this problem can be traced back to a CARE (continuous-time algebraic Riccati equation):

$$\boxed{KA + A^TK - KBR^{-1}B^TK + Q = 0}$$

The system can be stabilized with the $u = -Gx$ state feedback, where

$$\boxed{G = R^{-1}B^TK}$$

Example 9. Design an optimal LQR controller for the following system: $\dot{x} = 2x + u$, i.e $A = 2, B = 1$.
Solution. We minimize the following functional:

$$J = \frac{1}{2}\int 5x^2 + u^2 dt$$

meaning that in our case $Q = 5$ and $R = 1$. In this case (first order system – only one single state variable) the CARE will have the following form:

$$-K^2 + 4K + 5 = 0$$

The solutions for K are 5 and -1. By definition, we should choose the positive one, otherwise, we obtain a positive feedback.

$$G = 1 \cdot 1 \cdot 5 = 5$$

Finally, the computed state feedback: $u = -5x$.