Computer controlled systems

Lecture 7, March 31, 2017

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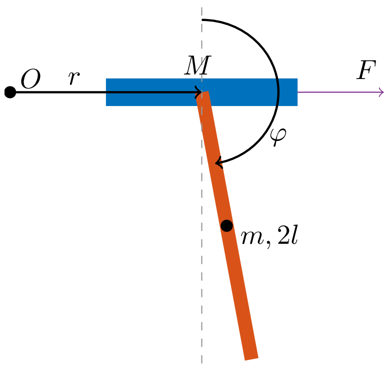
Exercises

We consider a simple pendulum mounted an a cart that can move horizontally:



M	is the mass of the cart
m	is the mass of the pendulum
$2l$	is the length of the pendulum
l	is the distance of the pivot point from the pendulum's center of mass
F	is an external force (input) acting on the cart
b	is the damping factor
r	is the (horizontal) position of the cart
$\dot{r} = v$	is the (horizontal) velocity of the cart
φ	is the angle of the cart (clockwise direction)
$\dot{\varphi} = \omega$	is the angular velocity of the cart (clockwise direction)
$\varphi = 0$	**unstable equilibrium point**: if the pendulum's center of mass is exactly **above** its pivot point (is vertical and pointing towards the sky)
$\varphi = \pi$	**stable equilibrium point**: if the pendulum's center of mass is exactly **below** its pivot point

This system has a nonlinear equation, which can be linearized in a certain operating point[1] (see Appendix). The state vector of the system is the following: $x = \begin{pmatrix} r & v & \varphi & \omega \end{pmatrix}^T$, furthermore, the external force F constitutes the input of the system (u). The nonlinear model of the system is: $\dot{x} = f(x) + g(x)u$, where

$$f(x) = \begin{pmatrix} v \\ \frac{1}{q}\Big(4ml\sin(\varphi)\omega^2 - 1.5mg\sin(2\varphi) - 4bv\Big) \\ \omega \\ \frac{3}{lq}\Big(-\frac{ml}{2}\sin(2\varphi)\omega^2 + (M+m)g\sin(\varphi) + b\cos(\varphi)v\Big) \end{pmatrix}, \quad g(x) = \frac{1}{lq}\begin{pmatrix} 0 \\ 4l \\ 0 \\ -3\cos(\varphi) \end{pmatrix} \tag{1}$$

where $q = 4(M+m) - 3m\cos(\varphi)^2$. For the full derivation see Appendix. For each exercise, you can use your own parameter configuration. Some examples are listed below.

(A) no friction	**(B)** with friction	**(C)** with friction + heavy rod
$M =$ 0.5 [kg]	$M =$ 0.5 [kg]	$M =$ 0.5 [kg]
$m =$ 0.2 [kg]	$m =$ 0.2 [kg]	$m =$ 10 [kg]
$l =$ 1 [m]	$l =$ 1 [m]	$l =$ 1 [m]
$g =$ 9.8 [m/s^2]	$g =$ 9.8 [m/s^2]	$g =$ 9.8 [m/s^2]
$b =$ 0 [kg/s]	$b =$ 10 [kg/s]	$b =$ 10 [kg/s]

[1] munkapont

1. Linearized model around the *stable* equilibrium point ($\varphi = \pi$)

Linearized model around the operating point $x^* = \begin{pmatrix} 0 & 0 & \pi & 0 \end{pmatrix}^T$:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{4b}{4M+m} & -\frac{3mg}{4M+m} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{3b}{l(4M+m)} & -\frac{3(M+m)g}{l(4M+m)} & 0 \end{pmatrix}, \quad B = \frac{1}{l(4M+m)} \begin{pmatrix} 0 \\ 4l \\ 0 \\ 3 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{2}$$

(`ss,tf`) 1. Determine the system's transfer function:

$$H(s) = \begin{pmatrix} H_{u\to r}(s) \\ H_{u\to\varphi}(s) \end{pmatrix} \tag{3}$$

(`impulse`) 2. Determine the impulse response of the system

(`step`) 3. Determine the step response of the system for both $H_{u\to r}(s)$ and $H_{u\to\varphi}(s)$. Determine the DC gain of the system.

(`eig`) 4. Determine the poles of the system. Is the linearized model locally/globally/asymptotically stable? What can we say about the original nonlinear system's stability? How does the stability properties change if we assume friction?

(`bodeplot`) 5. Determine the Bode plot of the transfer function $H_{u\to\varphi}(s)$. Set the frequency unit to be in Hz. Determine the own (or resonance) frequency (f_r) of the system.

(`nyquist`) 6. Plot the Nyquist diagram of $H_{u\to\varphi}(s)$.

(`lsim`) 7. Plot the output of the system if the input is $u_i(t) = A_i \sin(2\pi f_i t)$, where

(a) $f_2 = f_r\,[Hz],\ A_2 = 1\,[N]$ (b) $f_3 = 4\,[Hz],\ A_3 = 20\,[N]$ (c) $f_1 = 0.1\,[Hz],\ A_1 = 1\,[N]$

Considering the Bode diagram, what is expected to happen in each cases? In certain cases, we shall notice that the system's motion is quite unusual, why?

(`ode45`) 8. Solve the linearized differential equation $\dot{x} = Ax + Bu$ with different initial conditions. The input may be zero first, than you can use the values from the previous example.

(`ctrb`) 9. Is the linearized model controllable?

(`obsv`) 10. Is the linearized model observable? How does this change if we measure only the angle of the rod φ.

(`null`) (a) Compute the kernel (null space) of $\mathcal{O}_4$.

(`orth`) (b) Give the bases of the image space of $\mathcal{O}_4$.

(c) Give the matrix T of the linear state transformation, which produces the observability staircase representation:

$$\begin{pmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{pmatrix} = \begin{pmatrix} A_o & 0 \\ A_{21} & A_{\bar{o}} \end{pmatrix} \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix} + \begin{pmatrix} B_o \\ B_{\bar{o}} \end{pmatrix} u$$
$$y = \begin{pmatrix} C_o & 0 \end{pmatrix} \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix}$$

2. Nonlinear system simulation

11. Solve the nonlinear ODE (1) numerically, use the `ode45` solver:

(a) $x_0 = \begin{pmatrix} 0 & 0 & \frac{5\pi}{6} & 0 \end{pmatrix}^T$, $u(t) = 0$

(b) $x_0 = \begin{pmatrix} 0 & 0 & \frac{\pi}{6} & 0 \end{pmatrix}^T$, $u(t) = 0$

(c) $x_0 = 0$, $u(t) = \sin(2\pi f_r t)$

(d) You can play with x_0 and $u(t)$ as you want

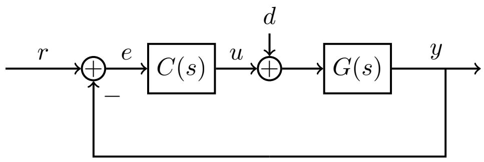
3. PID controller design

12. Consider the following SISO model given by the transfer function:

$$G(s) = \frac{s^2+3s+2}{s^3+2s^2-6s+8} \tag{4}$$

(`pzmap`) (a) Determine the poles and the zeros of the system. Is the system minimum-phase?

(`pidTuner`) (b) Design a PID controller $C(s)$ which provides stability and reference tracking.



$$C(s) = K_p + \frac{K_i}{s} + K_d s = \frac{K_d s^2 + K_p s + K_i}{s} \tag{5}$$

Appendix

I. Linearize a nonlinear model around an equilibrium point

We have a nonlinear system in the following form:

$$\dot{x} = F(x,u) = f(x) + g(x)u \tag{6}$$

Let $x^* \in \mathbb{R}^n$ be an equilibrium point of the nonlinear system, which means that $F(x^*,0) = f(x^*) = 0$. We assume that the system operates around this equilibrium point, and by default there is no input given to the system. Therefore, we say that the system's operating point[2] is $(x^*, u^* = 0)$.

The Jacobian matrix of $F(x,u)$ is

$$\mathrm{D}[F(x,u)] = \left(\tfrac{\partial F(x,u)}{\partial x} \;\middle|\; \tfrac{\partial F(x,u)}{\partial u} \right) = \left(\tfrac{\partial f(x)}{\partial x} + \tfrac{\partial g(x)}{\partial x} u \;\middle|\; g(x) \right) \tag{7}$$

The value of the Jacobian matrix in this operating point is

$$\mathrm{D}[F(x^*,0)] = \left(\tfrac{\partial f(x^*)}{\partial x} \;\middle|\; g(x^*) \right) \tag{8}$$

Now we estimate $F(x,u)$ by its first order Taylor polynomial around the operating point:

$$\begin{aligned} F(x,u) &\simeq \underbrace{F(x^*,0)}_{0} + \mathrm{D}[F(x^*,0)] \begin{pmatrix} x - x^* \\ u - 0 \end{pmatrix} \\ F(x,u) &\simeq \frac{\partial f(x^*)}{\partial x}(x - x^*) + g(x^*)u \end{aligned} \tag{9}$$

Hence, the linear model is

$$\dot{x} = A(x - x^*) + Bu, \quad \text{where} \quad \begin{aligned} A &:= \frac{\partial f(x^*)}{\partial x} \\ B &:= g(x^*) \end{aligned} \tag{10}$$

There's only one more thing left, we need to center the system. We introduce the centered state vector $\bar{x} := x - x^*$. Therefore, the time derivative of the transformed state vector will be:

$$\dot{\bar{x}} = \dot{x} = A(x - x^*) + Bu = A\bar{x} + Bu \tag{11}$$

Finally, we obtained the centered linearized model:

$$\dot{\bar{x}} = A\bar{x} + Bu, \quad \text{where} \quad \begin{aligned} A &:= \frac{\partial f(x^*)}{\partial x} \\ B &:= g(x^*) \end{aligned} \tag{12}$$

[2] munkapont

II. Derivation of the inverted pendulum's equation

The equation of the inverted pendulum is the following:

$$\begin{aligned} (M+m)\ddot{x} + ml\ddot{\varphi}\cos(\varphi) - ml\dot{\varphi}^2\sin(\varphi) &= F \\ ml\ddot{x}\cos(\varphi) + \frac{4}{3}ml^2\ddot{\varphi} - mgl\sin(\varphi) &= 0 \end{aligned} \tag{13}$$

The nonlinear state space equation of the inverted pendulum:

$$\begin{cases} \dot{x} = v \\ \dot{v} = \dfrac{1}{q}\big(4ml\sin(\varphi)\omega^2 - 1.5mg\sin(2\varphi) - 4bv\big) + \dfrac{4}{q}F \\ \dot{\varphi} = \omega \\ \dot{\omega} = \dfrac{3}{lq}\left(-\dfrac{ml}{2}\sin(2\varphi)\omega^2 + (M+m)g\sin(\varphi) + b\cos(\varphi)v\right) - \dfrac{3\cos(\varphi)}{lq}F \end{cases} \tag{14}$$

where $q = 4(M+m) - 3m\cos(\varphi)^2$. Let the state vector be $x = \begin{pmatrix} x & v & \varphi & \omega \end{pmatrix}^T$.

$$f(x) = \begin{pmatrix} v \\ \frac{1}{q}\Big(4ml\sin(\varphi)\omega^2 - 1.5mg\sin(2\varphi) - 4bv\Big) \\ \omega \\ \frac{3}{lq}\Big(-\frac{ml}{2}\sin(2\varphi)\omega^2 + (M+m)g\sin(\varphi) + b\cos(\varphi)v\Big) \end{pmatrix}, \quad g(x) = \frac{1}{lq}\begin{pmatrix} 0 \\ 4l \\ 0 \\ -3\cos(\varphi) \end{pmatrix} \tag{15}$$

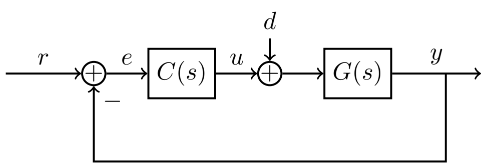
Linearized model around the stable operating point $x^* = \begin{pmatrix} 0 & 0 & \pi & 0 \end{pmatrix}^T$:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{4b}{4M+m} & -\frac{3mg}{4M+m} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{3b}{l(4M+m)} & -\frac{3(M+m)g}{l(4M+m)} & 0 \end{pmatrix}, \quad B = \frac{1}{l(4M+m)}\begin{pmatrix} 0 \\ 4l \\ 0 \\ 3 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{16}$$

Linearized state space model around the unstable operating point $x^* = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}^T$ is:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{4b}{4M+m} & -\frac{3mg}{4M+m} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{3b}{l(4M+m)} & \frac{3(M+m)g}{l(4M+m)} & 0 \end{pmatrix}, \quad B = \frac{1}{l(4M+m)}\begin{pmatrix} 0 \\ 4l \\ 0 \\ -3 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{17}$$

III. A simple control loop (SISO)



- r reference input
- d input disturbance (eg. wind, noise, fault of the actuator, etc.)
- u control input computed by the controller $C(s)$
- y output of system $G(s)$
- e error: difference between the reference input r and the output y

We derive, how the reference input r and the input disturbance d influence the output of $G(s)$:

$$\begin{aligned} y &= G(s)(u+d) = G(s)(u + C(s)(r-y)) \\ &= G(s)d + G(s)C(s)r - G(s)C(s)y \\ & \boxed{y = \frac{G(s)}{1+G(s)C(s)}d + \frac{G(s)C(s)}{1+G(s)C(s)}r} \end{aligned} \tag{18}$$

In general an actuator[3] has a limited power, and it cannot perform arbitrarily large control input u. Therefore, during the controller design, we need to consider what would be the actual control input (u) determined by the controller $C(s)$. From the closed loop system, we can derive the transfer function

[3] eg. in case of the inverted pendulum the actuator could be the DC motor of cart

describing the influence of r and d on the control input u:

$$
\begin{aligned}
u &= C(s)(r - y) = C(s)(r - G(s)(d + u)) \\
&= C(s)r - C(s)G(s)d - C(s)G(s)u \\
&\boxed{u = \frac{C(s)}{1 + G(s)C(s)} r + \frac{-G(s)C(s)}{1 + G(s)C(s)} d}
\end{aligned}
\tag{19}
$$