Computer controlled systems

Lecture 4

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1 State space transformation

As we shall already know, the state space model is not unique. For the given example, define a new SSM using a state space transformation.

$$A = \begin{pmatrix} -6 & -4 \\ 2 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \quad C = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

Let the linear transformation of the state vector be the following:

$$\bar{x}_1 = x_1 + x_2$$
$$\bar{x}_2 = 3x_1 - 2x_2$$

In matrix form:

$$\begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 3 & -2 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$\bar{x} = Tx,\ x = T^{-1}\bar{x} \quad \rightarrow \quad$ state state space equation can be written for the new state vector $\bar{x}$ as well

$$\dot{x} = Ax + Bu \quad \rightarrow \quad T^{-1}\dot{\bar{x}} = AT^{-1}\bar{x} + Bu$$

$$\dot{\bar{x}} = TAT^{-1}\bar{x} + TBu \quad \rightarrow \quad \bar{A} = TAT^{-1} \quad \bar{B} = TB$$

$$y = Cx = CT^{-1}\bar{x} \quad \rightarrow \quad \bar{C} = CT^{-1}$$

Returning to the example:

$$T = \begin{pmatrix} 1 & 1 \\ 3 & -2 \end{pmatrix} \quad T^{-1} = -\frac{1}{5}\begin{pmatrix} -2 & -1 \\ -3 & 1 \end{pmatrix}$$

$$\bar{A} = TAT^{-1} = \begin{pmatrix} -4 & 0 \\ -16 & -2 \end{pmatrix} \quad \bar{B} = TB = \begin{pmatrix} 4 \\ 12 \end{pmatrix} \quad \bar{C} = CT^{-1} = \begin{pmatrix} \frac{3}{5} & -\frac{1}{5} \end{pmatrix}$$

If the original and the transformed SSM are (A, B, C) and $(\bar{A}, \bar{B}, \bar{C})$, respectively, determine the transformation matrix T, which connects them.

$$A = \begin{pmatrix} 3 & 2 \\ -4 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix} \tag{1}$$

$$\bar{A} = \begin{pmatrix} 1.8 & 1.6 \\ -4.4 & 2.2 \end{pmatrix} \quad \bar{B} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad \bar{C} = \begin{pmatrix} 0.4 & -0.2 \end{pmatrix} \tag{2}$$

Solution. $\bar{B} = TB,\ \bar{A}\bar{B} = TAB \quad \rightarrow \quad T \cdot [B|AB] = [\bar{B}|\bar{A}\bar{B}] \quad \rightarrow \quad \boxed{T = \bar{\mathcal{C}}_2 \cdot \mathcal{C}_2^{-1}}$, where $\mathcal{C}_2 = [B|AB]$ and $\bar{\mathcal{C}}_2 = [\bar{B}|\bar{A}\bar{B}]$ are the controllability matrices of (1) and (2), respectively.
Remark. B and AB are (2×1) matrices.

$$\mathcal{C}_2 = \begin{pmatrix} 1 & 5 \\ 1 & -3 \end{pmatrix}, \quad \mathcal{C}_2^{-1} = \frac{1}{-8}\begin{pmatrix} -3 & -5 \\ -1 & 1 \end{pmatrix}, \quad \bar{\mathcal{C}}_2 = \begin{pmatrix} 3 & 7 \\ 1 & -11 \end{pmatrix}$$

$$T = \frac{1}{-8} \cdot \begin{pmatrix} 3 & 7 \\ 1 & -11 \end{pmatrix} \cdot \begin{pmatrix} -3 & -5 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \qquad T^{-1} = \frac{-1}{5} \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$$

Just as in the previous example, determine the transformation matrix T.

$$A = \begin{pmatrix} 2 & 0 \\ -1 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix} \tag{3}$$

$$\bar{A} = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} \quad \bar{B} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \bar{C} = \begin{pmatrix} \frac{-1}{2} & \frac{1}{2} \end{pmatrix} \tag{4}$$

Solution. $T = \begin{pmatrix} -1 & 2 \\ 1 & 2 \end{pmatrix} \quad T^{-1} = \frac{-1}{4} \cdot \begin{pmatrix} 2 & -2 \\ -1 & -1 \end{pmatrix}$

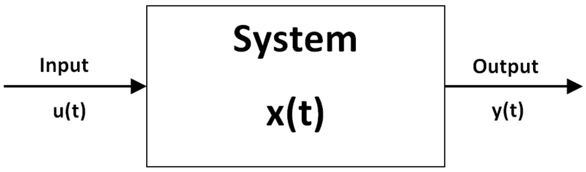
Remark. In case of SISO model, this method can be applied for an even higher dimensional state-space model, but then the controllability matrix will involve further rows. If the state vector is n-dimensional ($A \in \mathbb{R}^{n\times n}$), than $\mathcal{C}_n = [B|AB|A^2B|\dots|A^{n-1}B]$. To conclude, if the SSM is controllable:

$$\boxed{T = \bar{\mathcal{C}}_n \cdot \mathcal{C}_n^{-1}} \tag{5}$$

Megjegyzés: SISO modell esetén a fenti módszer több állapotváltozó esetén is alkalmazható, de ekkor több oszlopra van szükség. Ha $A \in \mathbb{R}^{n\times n}$, akkor a $[B|AB|A^2B|\dots|A^{n-1}B]$ alakú mátrixokkal lehet számolni.

2 Controllability, observability

In general Given the following CT-LTI system: The question arouse: In the full knowledge of $y(t)$ and



$u(t)$ can we say something about the unknown state vector $x(t)$? In the other words is $x(t)$ **observable**?

The second question would be the following: is there an input function $u(t)$, with which we can lead the system from the initial state x_0 to state x_1 in a finite time. If we can do so (for every possible initial and final states), we say that the system is **controllable**.

2.1 Observability

Theorem 1. Sufficient and necessary condition for observability

A state space model described by matrices (A, B, C) is observable if and only if (iff) its observability matrix $\mathcal{O}_n$ is full-rank:

$$\mathcal{O}_n = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}, \qquad \operatorname{rank}(\mathcal{O}_n) = n$$

Remark. In SISO case $\mathcal{O}_n$ is a square matrix, which is full-rank iff its determinant is nonzero.

Example 1. Is the system (A, B, C) observable?

$$A = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad C = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

The observability matrix is the following

$$CA = \begin{pmatrix} 2 & 1 \end{pmatrix} \to \mathcal{O}_2 = \begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}, \quad \det(\mathcal{O}_2) = -2 \neq 0 \Rightarrow \mathcal{O}_2 \text{ is full-rank}$$

Hence, $x(t)$ is observable, namely, using $y(t)$ and its time derivetive $\dot{y}(t)$, we can compute the actual value of $x(t)$

$$\begin{cases} y(t) = Cx(t) \\ \dot{y}(t) = CAx(t) + CBu(t) \end{cases} \Rightarrow x(t) = \mathcal{O}_2^{-1} \begin{pmatrix} y(t) \\ \dot{y}(t) - CBu(t) \end{pmatrix} \tag{6}$$

Example 2. Unobservable subspace (mathematical background presented in B.1)

Given the state space model:

$$A = \begin{pmatrix} 1 & 2 \\ -2 & -3 \end{pmatrix}, \quad B : \text{arbitrary}, \quad C = \begin{pmatrix} 1 & 1 \end{pmatrix}, \quad \mathcal{O}_n = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \tag{7}$$

A basis for the kernel of $\mathcal{O}_n$ is $v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. This means that

→ if there is a zero input and $x(0) = \lambda v_1 \in \mathcal{O}_2$, than $x(t) \in \text{Ker}(\mathcal{O}_2)$ (Proposition 9) and $y(t) = 0$ for every $t > 0$.

→ for a given input $u(t)$ and with an initial condition $x(0) = x_0 + \lambda v_1 \in x_0 + \text{Ker}(\mathcal{O}_2)$ (where $\lambda \in \mathbb{R}$ is arbitrary) the system will produce *the same ouput* $y(t)$.

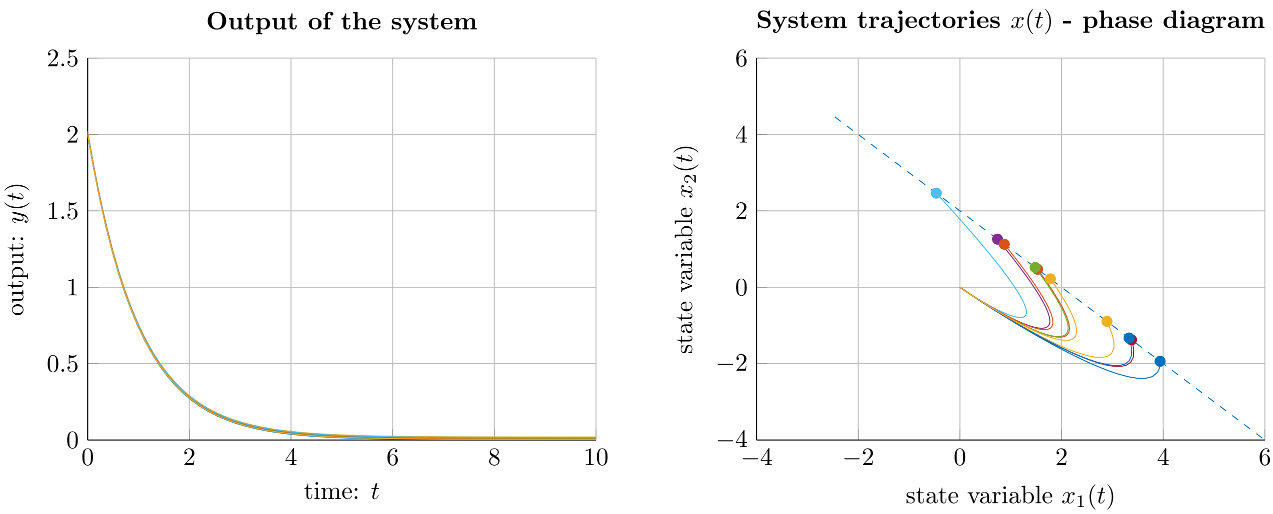


Figure 1. Simulation of system (7) from different initial conditions $x(0) \in x_0 + \text{Ker}(\mathcal{O}_2)$ (denoted by dots) with zero input. As one can observe, the state trajectories are different, however this difference does not appear in the output of the system. In this example $u \equiv 0$ and $x_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. The blue dashed line in the right figure illustrates the actual unobservability subspace of the system corresponding to x_0.

2.2 Controllability

Given a strictly proper state space model (A, B, C) with $x(t_0)$ initial and $x(t_1) \neq x(t_0)$ final condition. The question arises, is there any input function $u(t)$, which leads the system from $x(t_0)$ to $x(t_1)$ in a finite time.

Theorem 2. Controllability

A state space model described by matrices (A, B, C) is controllable iff its controllability matrix $\mathcal{C}_n$ is full-rank:

$$\mathcal{C}_n = \begin{pmatrix} B & AB & \cdots & A^{n-1}B \end{pmatrix}, \quad \text{rank}(\mathcal{C}_n) = n$$

Remark. In SISO case $\mathcal{C}_n$ is a square matrix, which is full-rank iff its determinant is nonzero.

Example 3.

$$A = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 \end{pmatrix}, \qquad \mathcal{C}_2 = \begin{pmatrix} B & AB \end{pmatrix} = \begin{pmatrix} 1 & 5 \\ 1 & 3 \end{pmatrix}$$

This system is controllable, since the determinant of $\mathcal{C}_2$ is nonzero. In this case the controllability subspace is the whole $\mathbb{R}^2$ itself. If we start the system from zero initial condition, we can lead the system (with an appropriate input) to any other states of the controllability subspace in a finite time.

Example 4. Controllable subspace (mathematical background presented in B.2)

Given the following state space system and its rank-deficient controllability matrix:

$$A = \begin{pmatrix} -1 & 2 & -2 \\ -\frac{2}{3} & -6 & \frac{20}{3} \\ -\frac{1}{2} & -1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 8 \\ 0 \end{pmatrix}, \quad \text{eigenvalues of } A\text{: } \begin{pmatrix} -2 \\ -2 \\ -4 \end{pmatrix}, \quad \mathcal{C}_3 = \begin{pmatrix} 0 & 16 & -96 \\ 8 & -48 & 224 \\ 0 & -8 & 48 \end{pmatrix} \tag{8}$$

The basis vectors of $\text{Im}(\mathcal{C}_3)$ are: $v_1 = \begin{pmatrix} 0.3832 \\ -0.9036 \\ -0.1916 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0.8082 \\ 0.4285 \\ -0.4041 \end{pmatrix}$. They span a 2-dimensional subspace in $\mathbb{R}^3$, illustrated by the green plane in the Figure 2. If we start the system from an initial condition which is an element of this subspace $x(0) \in \text{Im}(\mathcal{C}_3)$, the system trajectory will never leave this subspace. If the initial condition is outside of $\text{Im}(\mathcal{C}_3)$ and A is stable, the system trajectory will tend to this subspace.

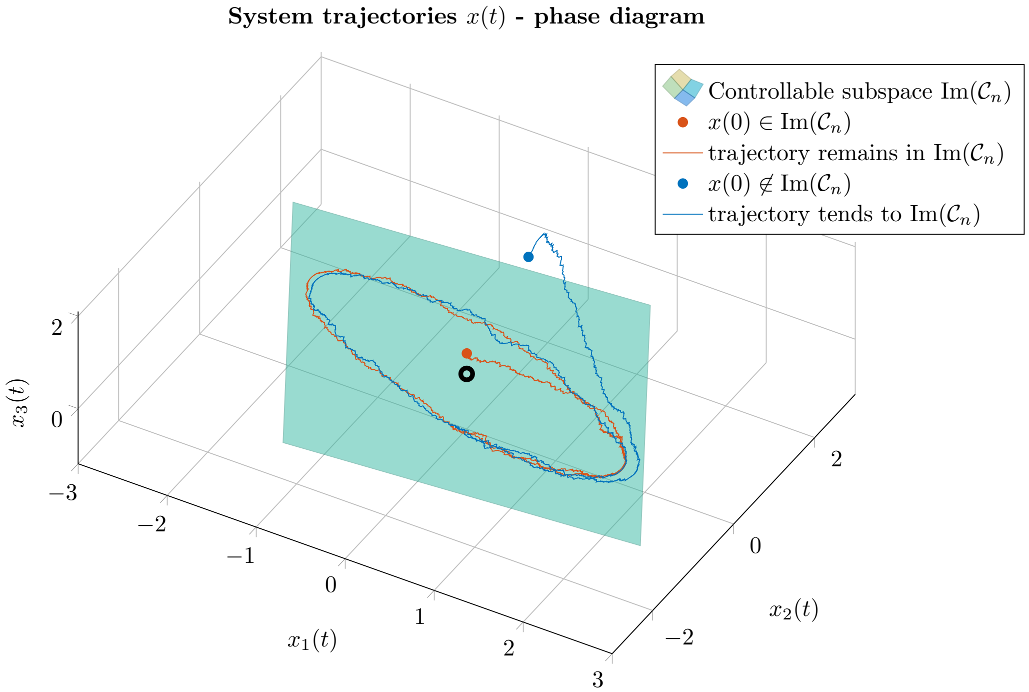


Figure 2. Simulation of system (7) from different initial conditions

Example 5.

Compute the controllable subspace of $\dot{x} = Ax + Bu$, where

$$A = \begin{pmatrix} 1 & 2 & -2 \\ -0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \; B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \tag{9}$$

To check your solutions, we give:

$$A^2 = \begin{pmatrix} -1 & 4 & -4 \\ 0 & 1 & 0 \\ 2 & 2 & -1 \end{pmatrix}, \; \mathcal{O}_3 = \begin{pmatrix} -1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & -2 \end{pmatrix}. \tag{10}$$

2.3 Controllability and observability in case of a diagonal SSM

$$A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \quad B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \quad AB = \begin{pmatrix} a_1 b_1 \\ a_2 b_2 \end{pmatrix} \quad C = \begin{pmatrix} c_1 & c_2 \end{pmatrix} \quad CA = \begin{pmatrix} c_1 a_1 & c_2 a_2 \end{pmatrix}$$

$$\mathcal{C}_2 = \begin{pmatrix} b_1 & a_1 b_1 \\ b_2 & a_2 b_2 \end{pmatrix} \qquad \mathcal{O}_2 = \begin{pmatrix} c_1 & c_2 \\ c_1 a_1 & c_2 a_2 \end{pmatrix}$$

SISO rendszer diagonális A mátrix esetén
irányítható $\iff$ a fátlóbeli elemek páronként különbözek, és $\forall i \; b_i \neq 0$
megfigyelhet $\iff$ a fátlóbeli elemek páronként különbözek, és $\forall j \; c_j \neq 0$

Theorem 3. The rank of $\mathcal{O}_n$ and $\mathcal{C}_n$ is invariant to the state space transformations.

Proof.

$$\bar{A} = TAT^{-1} \quad \bar{B} = TB \quad \bar{C} = CT^{-1}$$

$$\bar{\mathcal{C}}_n = \begin{pmatrix} TB & TAT^{-1}TB \end{pmatrix} = T \begin{pmatrix} B & AB \end{pmatrix} = T\mathcal{C}_n$$

$$\bar{\mathcal{O}}_n = \begin{pmatrix} CT^{-1} \\ CT^{-1}TAT^{-1} \end{pmatrix} = \begin{pmatrix} C \\ CA \end{pmatrix} T^{-1} = \mathcal{O}_n T^{-1}$$

□

2.4 Markov parameters

$$CA^iB$$

Markov parameters are invariant to the state space transformations.

$$\bar{C}\bar{B} = CT^{-1}TB = CB$$

$$\bar{C}\bar{A}\bar{B} = CT^{-1}TAT^{-1}TB = CAB$$

3 Joint controllability and observability

- Egy $H(s) = \frac{b(s)}{a(s)}$ (SISO) átviteli függvény n-edrend realizációjának nevezzük az (A, B, C, D) állapottér-modellt, ha $H(s) = C(sI - A)^{-1}B + D$, ahol $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times 1}$, $C \in \mathbb{R}^{1 \times n}$, $D \in \mathbb{R}$ (nem egyértelm!)
- Egy $H(s)$ átviteli függvény n-edrend realizációját minimálisnak nevezzük, ha nem létezik nála kisebb rend realizáció.

- Egy n-dimenziós (A, B, C, D) állapottér-modellt együttesen irányíthatónak és megfigyelhetnek nevezünk, ha teljesülnek rá az irányíthatóság és a megfigyelhetség feltételei (azaz $\mathcal{O}_n$ és $\mathcal{C}_n$ teljes rangú).
- Egy ÁTM minimális $\iff$ egyszerre irányítható és megfigyelhet.

Example 6. Is the state space representation minimal?

$$A = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad C = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

Transfer function: $H(s) = \frac{s}{s^2-3s-4}$. This SSM is minimal, since $H(s)$ is irreducible and the degree of the denominator is equal to the order of the state space realization ($n = 2$).

Example 7. Is the state space representation minimal?

$$A = \begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 1 \end{pmatrix}$$

$$H(s) = C(sI - A)^{-1}B = \frac{s+1}{s^2+4s+3} = \frac{s+1}{(s+1)(s+3)}$$

This SSM is not minimal, meaning the one of two properties is broken: the SSM is controllable but its is no observable.

Example 8. Is the state space representation minimal?

$$A = \begin{pmatrix} -6 & -4 \\ 2 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \quad C = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

Controllability matrix:

$$\mathcal{C}_2 = \begin{pmatrix} B & AB \end{pmatrix} = \begin{pmatrix} 4 & -24 \\ 0 & 8 \end{pmatrix}$$

The determinant of matrix $\mathcal{C}_2$ is nonzero, therefore, it is controllable.
Observability matrix:

$$\mathcal{O}_2 = \begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$$

The determinant of matrix $\mathcal{O}_2$ is nonzero, therefore, it is observable. Consequently, the SSM is minimal.

Example 9. (MIMO case) Is the state space representation minimal?

$$A = \begin{pmatrix} 1 & 4 \\ -2 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 4 & 1 \\ 2 & 3 & 0 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 2 \\ 0 & 7 \end{pmatrix}$$

$$AB = \begin{pmatrix} 9 & 16 & 1 \\ 2 & -2 & -2 \end{pmatrix} \quad CA = \begin{pmatrix} -3 & 8 \\ -14 & 14 \end{pmatrix}$$

$$\mathcal{O}_2 = \begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 7 \\ -3 & 8 \\ -14 & 14 \end{pmatrix} \quad \mathcal{C}_2 = \begin{pmatrix} B & AB \end{pmatrix} = \begin{pmatrix} 1 & 4 & 1 & 9 & 16 & 1 \\ 2 & 3 & 0 & 2 & -2 & -2 \end{pmatrix}$$

Matrix $\mathcal{O}_2$ is full-column-rank, and $\mathcal{C}_2$ is full row rank, meaning that the system is jointly controllable and observable and (A, B, C) is minimal.

Example 10. Is the SSM minimal? If not give a minimal representation.

$$A = \begin{pmatrix} -3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} \quad C = \begin{pmatrix} 3 & 0 & 4 \end{pmatrix}$$

$$\begin{aligned} H(s) &= \sum_{i=1}^{n} \frac{c_i b_i}{s - \lambda_i} = \frac{3 \cdot 1}{s+3} + \frac{0 \cdot 2}{s-4} + \frac{6 \cdot 4}{s-6} = \frac{3(s-6) + 24(s+3)}{(s+3)(s-6)} \\ H(s) &= \frac{27s + 54}{s^2 - 3s - 18} \end{aligned}$$

The SSM is not minimal, because the transfer function can be reduced.

$$A^2 = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 36 \end{pmatrix} \qquad \mathcal{O}_n = \begin{pmatrix} 3 & 0 & 4 \\ -9 & 0 & 24 \\ 27 & 0 & 144 \end{pmatrix} \qquad \mathcal{C}_n = \begin{pmatrix} 1 & -3 & 9 \\ 2 & 8 & 32 \\ 6 & 36 & 216 \end{pmatrix}$$

A minimal SSM can be given by skipping the single degenerated state variable:

$$A = \begin{pmatrix} -3 & 0 \\ 0 & 6 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 6 \end{pmatrix} \quad C = \begin{pmatrix} 3 & 4 \end{pmatrix}$$

A minimal realization can also be given using the controller form:

$$A = \begin{pmatrix} 3 & 18 \\ 1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad C = \begin{pmatrix} 27 & 54 \end{pmatrix}$$

Example 11. It is given a SSM in the controller form. Is the SSM jointly controllable and observable?

$$A = \begin{pmatrix} 0 & 7 & -6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad C = \begin{pmatrix} 0 & 3 & 9 \end{pmatrix}$$

Transfer function:

$$H(s) = \frac{3s + 9}{s^3 - 7s + 6}$$

The realization is most be controllable, since it is given in controller form:

$$A^2 = \begin{pmatrix} 7 & -6 & 0 \\ 0 & 7 & -6 \\ 1 & 0 & 0 \end{pmatrix} \qquad \mathcal{O}_n = \begin{pmatrix} 0 & 3 & 9 \\ 3 & 9 & 0 \\ 9 & 21 & -18 \end{pmatrix} \qquad \mathcal{C}_n = \begin{pmatrix} 1 & 0 & 7 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{matrix} \text{rank}(\mathcal{C}_n) = 3 \\ \text{rank}(\mathcal{O}_n) = 2 \end{matrix}$$

However the SSM is not observable, because it is not minimal: $H(s)$ is reducible by $s + 3$. Using the controller form (on the irreducible form of $H(s)$), we can obtain a jointly controllable and observable realization Tehát nem együttesen megfigyelhet és irányítható a rendszer. The a unobservable subspace

$$\text{Ker}(\mathcal{O}_n) = \left\{ \alpha \begin{pmatrix} 9 \\ -3 \\ 1 \end{pmatrix} \,\middle|\, \alpha \in \mathbb{R} \right\}$$

Felhasználás: Állapotmegfigyelk tervezése

Bizonyos mennyiségeket (pl. szögsebesség) nem tudunk mérni, csak becsülni. Ld.: 3. ábra

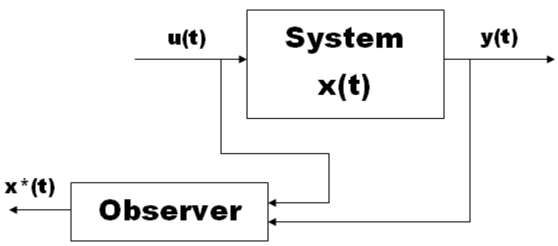


Figure 3. State observer design

References

[1] Alexey Grigorev. The Fundamental Theorem of Linear Algebra. Technische Universität Berlin.

[2] Lantos Béla. *Irányítási rendszerek elmélete és tervezése I.* Akadémiai Kiadó Budapest, 2001.

[3] A. D. Lewis. *A Mathematical Approach to Classical Control*. 2003.

A Supplementary material in linear algebra (not needed for the exam)

Theorem 4. The fundamental theorem of linear algebra

Let $\mathcal{A} : \mathbb{R}^n \to \mathbb{R}^m$, $\mathcal{A}(x) = Ax$, where $A \in \mathbb{R}^{m\times n}$. Then the followings are true

$$\mathrm{Im}(A) = \mathrm{Ker}\big(A^T\big)^{\perp} \subset \mathbb{R}^m \tag{11a}$$

$$\mathrm{Im}\big(A^T\big) = \mathrm{Ker}(A)^{\perp} \subset \mathbb{R}^n \tag{11b}$$

Furthermore

$$\mathrm{Im}(A) \otimes \mathrm{Ker}\big(A^T\big) = \mathbb{R}^m \tag{12a}$$

$$\mathrm{Im}\big(A^T\big) \otimes \mathrm{Ker}(A) = \mathbb{R}^n \tag{12b}$$

Remark. If $r = \mathrm{rank}(A)$, than

$$\dim \mathrm{Im}(A) = r, \qquad \dim \mathrm{Ker}\big(A^T\big) = m - r \tag{13a}$$

$$\dim \mathrm{Im}\big(A^T\big) = r, \qquad \dim \mathrm{Ker}(A) = n - r \tag{13b}$$

Proof. Proof of (11a) as presented in [1]. Let

$$A = \begin{pmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \dots & \boldsymbol{a}_n \end{pmatrix} \;\Rightarrow\; A^T = \begin{pmatrix} \boldsymbol{a}_1^T \\ \boldsymbol{a}_2^T \\ \vdots \\ \boldsymbol{a}_n^T \end{pmatrix} \tag{14a}$$

$$\boldsymbol{x} \in \mathrm{Ker}\big(A^T\big) \;\Rightarrow\; A^T\boldsymbol{x} = \begin{pmatrix} \boldsymbol{a}_1^T\boldsymbol{x} \\ \boldsymbol{a}_2^T\boldsymbol{x} \\ \vdots \\ \boldsymbol{a}_n^T\boldsymbol{x} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{14b}$$

$$\boldsymbol{y} \in \mathrm{Im}(A) \;\Rightarrow\; \exists \alpha_i \in \mathbb{R} \text{ such that } \boldsymbol{y} = \sum_{i=1}^{n} \alpha_i \boldsymbol{a}_i \tag{14c}$$

Note that $\boldsymbol{x}$ and $\boldsymbol{y}$ are arbitrary vector elements of $\mathrm{Ker}\big(A^T\big)$ and $\mathrm{Im}(A)$, respectively. Then we compute the dot product of $\boldsymbol{x}$ and $\boldsymbol{y}$:

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{y}^T\boldsymbol{x} = \sum_{i=1}^{n} \alpha_i \boldsymbol{a}_i^T \boldsymbol{x} = 0, \tag{15}$$

since $\boldsymbol{a}_i^T\boldsymbol{x} = 0$, $\forall i = \overline{1,n}$. Consequently, $\boldsymbol{x} \perp \boldsymbol{y}$ for all possible $x \in \mathrm{Ker}\big(A^T\big)$ and $y \in \mathrm{Im}(A)$, which means that the two subspaces are the ***orthogonal complement*** for each other:

$$\begin{aligned} \mathrm{Im}(A) &= \mathrm{Ker}\big(A^T\big)^{\perp} \\ \mathrm{Im}(A) \cap \mathrm{Ker}\big(A^T\big) &= \{0\} \end{aligned} \tag{16}$$

Following this idea, we can conclude that the subspaces are linearly independent, therefore,

$$\dim\Big(\mathrm{Im}(A) \otimes \mathrm{Ker}\big(A^T\big)\Big) = r + (m - r) = m. \tag{17}$$

This can only happend if ***direct product*** of the two spaces is $\mathbb{R}^m$, which completes the proof for (12a).

□

Proposition 5. (Self-adjoint operator) Let $\mathcal{A} : \mathbb{R}^n \to \mathbb{R}^n$, $\mathcal{A}(x) = Ax$, where $A \in \mathbb{R}^{n\times n}$ is a symmetric matrix: $A = A^T$. Than, as a consequence of Theorem 4, we have that

$$\mathrm{Im}(A) = \mathrm{Ker}(A)^{\perp} \text{ and } \mathrm{Im}(A) \otimes \mathrm{Ker}(A) = \mathbb{R}^n.$$

For more, see [2, Eq. (10.3)].

Proposition 6. Singular value decomposition (SVD)

If we make the SVD for matrix $A \in \mathbb{R}^{m\times n}$

$$A = U\Sigma V^T, \tag{18}$$

where

$$U \in \mathbb{R}^{m\times m} \text{ is unitary: } U^*U = I_m \tag{19a}$$
$$V \in \mathbb{R}^{n\times n} \text{ is unitary: } V^*V = I_n \tag{19b}$$
$$\Sigma \in \mathbb{R}^{m\times n} \text{ eigenvalues in the diagonal.} \tag{19c}$$

After this decomposition, the basis of the four subspaces (12) can be obtained as presented below.

$$\begin{aligned} \mathrm{Im}(A): &\quad \text{the first } r \text{ colums of } U \\ \mathrm{Ker}\big(A^T\big): &\quad \text{the last } m-r \text{ columns of } U \\ \mathrm{Im}\big(A^T\big): &\quad \text{the first } r \text{ columns of } V \\ \mathrm{Ker}(A): &\quad \text{the last } n-r \text{ columns of } V \end{aligned}$$

In short

$$\text{“ } A = \big[\mathrm{Im}(A) \quad \mathrm{Ker}\big(A^T\big)\big]\, \Sigma\, \big[\mathrm{Im}\big(A^T\big) \quad \mathrm{Ker}(A)\big]^T \text{ ”} \tag{20}$$

B Subspaces of the state space

Having a strictly proper $(D = 0)$ MIMO LTI system:

$$\begin{aligned} \dot{x} &= Ax + By \\ y &= Cx \end{aligned} \tag{21}$$

The state space could be partitioned as follows:

$$X = X_{co} \otimes X_{c\bar{o}} \otimes X_{\bar{c}o} \otimes X_{\bar{c}\bar{o}} \tag{22}$$

where $X_{..}$ are pairwise orthogonal subspaces of the state space, in other words:

$$\begin{aligned} &X_{co} \perp X_{c\bar{o}}, \quad X_{co} \perp X_{\bar{c}o}, \quad X_{co} \perp X_{\bar{c}\bar{o}}, \\ &X_{c\bar{o}} \perp X_{\bar{c}o}, \quad X_{c\bar{o}} \perp X_{\bar{c}\bar{o}}, \quad X_{\bar{c}o} \perp X_{\bar{c}\bar{o}}. \end{aligned} \tag{23}$$

B.1 Unobservable subspace $X_{\bar{o}} = \mathrm{Ker}(\mathcal{O}_n)$. Observable subspace $X_o = X_{\bar{o}}^{\perp} = \mathrm{Im}\big(\mathcal{O}_n^T\big)$.

Lemma 7. Linear independence of the first k rows of $\mathcal{O}_n$

If $\mathrm{rank}\,(\mathcal{O}_n) = k \le n$, then the first k rows of $\mathcal{O}_n$ are linearly independent, and any further rows of it can be expressed as the linear combination of the first k rows.

Formally: $\forall i \in \mathbb{N}\ \exists \alpha \in \mathbb{R}^k$, that $CA^{k+i} = \alpha^T \mathcal{O}_k$, where $\mathcal{O}_k \in \mathbb{R}^{k\times n}$ is defined as $\mathcal{O}_k = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{k-1} \end{pmatrix}$.

Remark. $\mathbb{N} = \{0, 1, 2, ..\}$, $\mathbb{N}_+ := \mathbb{N}\backslash\{0\}$.

Proof. The proof is given in the following three steps:

(i) If $k = n$, the set of row vectors (also called as “covariant vectors”) $C, CA, .., CA^{n-1}$ constitutes a linearly independent (covariant) basis for vector space $\mathbb{R}^n$, which means that any other row vectors in $\mathbb{R}^n$ can be expressed by their linear combinations, the same as $CA^{n+i}, \forall i \in \mathbb{N}$ can be.

(ii) Let k be the first natural number, for which there exists $\alpha \in \mathbb{R}^k$ such that $CA^k = \alpha^T \mathcal{O}_k$. Then

CA^{k+1} can also be expressed by the covariant vectors of $\mathcal{O}_k$:

$$CA^{k+1} = \left(CA^k\right)A = \left(\sum_{j=1}^{k} \alpha_j CA^{j-1}\right)A = \sum_{j=1}^{k-1} \alpha_j CA^j + \alpha_k \sum_{j=1}^{k} \alpha_j CA^{j-1} \tag{24}$$

By induction, we have that for every every $i \in \mathbb{N}$ there exists $\alpha \in \mathbb{R}^k : CA^{k+i} = \alpha \mathcal{O}_k$.

(iii) As a consequence of (ii), we can state that if $\operatorname{rank}(\mathcal{O}_n) = k < n$, that the first k rows of $\mathcal{O}_n$ are linearly independent (i.e. $\operatorname{rank}(\mathcal{O}_k) = k$). □

Lemma 8. For every $v \in \operatorname{Im}(\mathcal{O}_n^T)$, we have that $A^T v \in \operatorname{Im}(\mathcal{O}_n^T)$. In this sense, the observable subspace $X_o = \operatorname{Im}(\mathcal{O}_n^T) = \operatorname{Ker}(\mathcal{O}_n)^\perp \subseteq \mathbb{R}^n$ of the state space $X = \mathbb{R}^n$ is invariant with respect to the linear transformation $\mathcal{A}'(v) = A^T v$, i.e. $\mathcal{A}'(X_o) = X_o$.

Proof. Let $a(s) = \det(sI - A) = a_0 + a_1 s + ... + a_n s^n$. Due to Cayley-Hamilton theorem, we have that

$$a(A) = 0 \;\Rightarrow\; A^n = \frac{1}{a_n}\left(a_0 I + a_1 A + ... + a_{n-1} A^{n-1}\right) \tag{25}$$

□

Proposition 9. $x(0) \in \operatorname{Ker}(\mathcal{O}_n)$ and $u(t) = 0 \;\Rightarrow\; y(t) = 0$

Let $\operatorname{rank}(\mathcal{O}_n) = k < n$. If $x_0 \in \operatorname{Ker}(\mathcal{O}_n)$ and $u \equiv 0$, than $y(t) = 0$ for every $t > 0$, i.e

$$x(t) = e^{At} x_0 \in \operatorname{Ker}(\mathcal{O}_n)$$

In other words, if there is no input signal ($u(t) = 0$) and the initial condition x_0 belongs to the unobservable subspace $\operatorname{Ker}(\mathcal{O}_n)$, than the state response of the system $x(t) = e^{At} x_0$ will remain in this subspace.

Proof. As a consequence of Proposition 7, we have that if $CA^k x_0 = 0$ for $k = \overline{0, n-1}$, than $CA^k x_0 = 0$ holds for every $k \in \mathbb{N}$. If we consider the Taylor expansion of matrix exponent e^{At}, we have:

$$CA^k e^{At} x_0 = \sum_{j=0}^{\infty} \frac{t^k}{k!} \cdot \underbrace{CA^{k+j} x_0}_{0} = 0 \;\; \forall k = \overline{0, n-1} \;\;\Rightarrow\;\; \mathcal{O}_n e^{At} x_0 = 0 \Leftrightarrow e^{At} x_0 \in \operatorname{Ker}(\mathcal{O}_n) \tag{26}$$

Consequently, for a given unobservable state space model (A,B,C,D) if we start the system from the unobservable subspace $x(0) \in \operatorname{Ker}(\mathcal{O}_n)$ and having a zero input ($u \equiv 0$) the output will be zero $y(t) = 0$, for every $t > 0$. □

Proposition 10. Same output for all initial state of an unobservable class

Let us denote $v_1, .., v_{n-k} \in \mathbb{R}^n$, $k < n$ the basis vectors of the null space of $\mathcal{O}_n$:

$$\operatorname{Ker}(\mathcal{O}_n) = \left\{ \alpha_1 v_1 + .. + \alpha_{n-k} v_{n-k} = \alpha^T N \;\middle|\; \alpha \in \mathbb{R}^{n-k} \right\}, \quad \text{where } N := \begin{pmatrix} v_1 & .. & v_{n-k} \end{pmatrix} \in \mathbb{R}^{n \times (n-k)}$$

Matrix N is called an *annihilator* of $\mathcal{O}_n$, since $\mathcal{O}_n N = 0_{n \times (n-k)}$. Now we introduce the following notations:

$$x_0 + \operatorname{Ker}(\mathcal{O}_n) := \left\{ x_0 + \alpha^T N \;\middle|\; \alpha \in \mathbb{R}^{n-k} \right\} \tag{27}$$

From any initial condition $x(0) \in x_0 + \operatorname{Ker}(\mathcal{O}_n)$ and for a given input $u(t)$, the system will produce the same output $y(t)$.

Proof. The explicit solution of the state space model is

$$y(t) = Ce^{At} x(0) + C \int_0^t e^{A(t-\tau)} B u(\tau) \mathrm{d}\tau \tag{28}$$

Considering an initial condition $x(0) = x_0 + \alpha^T N \in x_0 + \operatorname{Ker}(\mathcal{O}_n)$ with an arbitrary $\alpha \in \mathbb{R}^{n-k}$, and keeping in mind, that $\alpha^T N \in \mathrm{Ker}(\mathcal{O}_n)$ (i.e. $CA^i\alpha^T N = 0$ for all $i \in \mathbb{N}$) we obtain:

$$y(t) = Ce^{At}\big(x_0 + \alpha^T N\big) + C\int_0^t e^{A(t-\tau)}Bu(\tau)\mathrm{d}\tau = Ce^{At}x_0 + C\int_0^t e^{A(t-\tau)}Bu(\tau)\mathrm{d}\tau \tag{29}$$

Finally, we can observe that the expression for $y(t)$ does not depend on α. It depends only on the input $u(t)$ and on x_0, furthermore, for each x_0 we obtain different outputs, x_0 defines the unobservability class, that the system is actually in. If we can find a particular solution $x(t)$ for the (under-determined) linear equation system

$$\mathcal{Y}(t) = \mathcal{O}_n x(t) + \mathcal{T}\mathcal{U}(t) \qquad [\texttt{lec\_03.pdf}, \text{ pg. } 10/31] \tag{30}$$

we can determine the actual unobservability class of the system, but we have no further informations about the state vector itself. □

Remark. Set $x_0 + \mathrm{Ker}(\mathcal{O}_n)$ is not a subspace of $\mathbb{R}^n$, since many properties of the vector space broke (eg. does not have a unity element), however, it is a k dimensional manifold (sokaság) in vector space $\mathbb{R}^n$.

B.2 Controllable subspace $X_c = \mathrm{Im}(\mathcal{C}_n)$. Uncontrollable subspace $X_{\bar{c}} = X_c^{\perp} = \mathrm{Ker}\big(\mathcal{C}_n^T\big)$.

Lemma 11. If (A, B, C) is not controllable $\mathrm{rank}(\mathcal{C}_n) = k < n$, the first k columns of $\mathcal{C}_n$ are linearly independent.

Proof. Same as Lemma 7. □

Lemma 12. For every $v \in \mathrm{Im}(\mathcal{C}_n)$, vector $Av \in \mathrm{Im}(\mathcal{C}_n)$. In this sense, the controllable subspace $X_c = \mathrm{Im}(\mathcal{C}_n) \subseteq \mathbb{R}^n$ of the state space $X = \mathbb{R}^n$ is invariant with respect to the linear transformation $\mathcal{A}(v) = Av$, i.e. $\mathcal{A}(X_c) = X_c$.

Proof. Let $v \in X_c = \mathrm{span}\left\langle B, AB, ..., A^{n-1}B\right\rangle$, therefore, there exist real values $\alpha_1, ..., \alpha_n \in \mathbb{R}$, such that

$$v = \sum_{i=1}^{n} \alpha_i A^{i-1}B \;\Rightarrow\; Av = \sum_{i=1}^{n} \alpha_i A^i B. \tag{31}$$

It is obvious that $A^iB \in X_c$ for all $i = \overline{1, n-1}$, furthermore, due to Lemma 11, A^nB can be expressed as the linear combination of vectors A^{i-1}, B, $i = \overline{1, n}$. Finally, we have that $Av \in X_c$. □

Proposition 13. $x_0 \in \mathrm{Im}\left(\mathcal{C}_n\right) \Rightarrow x(t) \in \mathrm{Im}\left(\mathcal{C}_n\right)$

If the initial condition $x(0) = x_0$ belongs to the controllable subspace of the state space, than the solution $x(t)$ will also belong to it. Formally:

$$x_0 \in \mathrm{Im}(\mathcal{C}_n) \Rightarrow x(t) \in \mathrm{Im}(\mathcal{C}_n)\, \forall t \geq 0. \tag{32}$$

If the initial condition is not an element of $\mathrm{Im}(\mathcal{C}_n)$, but the system is stable, than the trajectory will tend exponentially to the controllable subspace of the state space, i.e.

$$A \prec 0 \Rightarrow x(t) \to \mathrm{Im}(\mathcal{C}_n) \tag{33}$$

Proof. If $x_0 \in \mathrm{Im}(\mathcal{C}_n) = X_c$, than

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)\mathrm{d}\tau = \sum_{k=0}^{\infty} \frac{t^k}{k!} \underbrace{A^k x_0}_{\in X_c} + \int_0^t \sum_{k=0}^{\infty} \frac{(t-\tau)^k}{k!} \underbrace{A^k B}_{\in X_c} u(\tau)\mathrm{d}\tau \in X_c. \tag{34}$$

If $x_0 \notin X_c$ but $A \prec 0$ (is negative definite), than

$$x(t) = \underbrace{e^{At}x_0}_{\to 0} + \underbrace{\int_0^t e^{A(t-\tau)}Bu(\tau)\mathrm{d}\tau}_{\in X_c} \to X_c. \tag{35}$$

So, the solution tends to the controllable subspace. □

Theorem 14. (Control the system to a given state) If the system is controllable, there exists an input

$$u(t) = -B^T e^{A^T(t_1-t)} P^{-1}(t_1)\big(e^{At_1}x_0 - x_1\big), \text{ where } P(t) = \int_0^t e^{A\tau}BB^Te^{A^T\tau}\mathrm{d}\tau, \quad t \in [0, t_1], \tag{36}$$

which leads the system from $x(0)$ to $x(t_1) = x_1$ in a finite time $t_1 < \infty$.

Proof. A proof for it can be found in [3, Theorem 2.21]. □

B.3 Controllability staircase form

Proposition 15. Controllability staircase form

We construct the following transformation matrix $T^{-1} = S = \big(v_1, ..., v_k, w_{k+1}, ..., w_n\big)$, where $[v] = [v_1, ..., v_k]$ is the orthonormal (ON) basis of $X_c = \mathrm{Im}(\mathcal{C}_n)$ and $[w] = [w_{k+1}, ..., w_n]$ is the ON basis of $X_{\bar{c}} = \mathrm{Im}(\mathcal{C}_n)^\perp = \mathrm{Ker}\big(\mathcal{C}_n^T\big)$. Then the transformed matrices will have the form:

$$\bar{A} = TAT^{-1} = \begin{pmatrix} \bar{A}_{11} & \bar{A}_{12} \\ 0_{(n-k)\times k} & \bar{A}_{22} \end{pmatrix} \tag{37a}$$

$$\bar{B} = TB = \begin{pmatrix} \bar{B}_1 \\ 0_{(n-k)\times 1} \end{pmatrix} \tag{37b}$$

Using SVD: $\mathcal{C}_n = U_c\Sigma_cV_c^T$, $S := U_c$

Proof. (For simplicity, only for SISO) Since X_c and $X_{\bar{c}}$ are orthogonal complement of each other (i.e. $X_c \otimes X_{\bar{c}} = \mathbb{R}^n$), $[v, w]$ is an ON basis of $\mathbb{R}^n$. In other words: S is an orthogonal matrix with the well-known properties:

$$S^TS = I_n \quad \Rightarrow \quad S^{-1} = S^T = \begin{pmatrix} V^T \\ W^T \end{pmatrix}, \quad \text{where } V = \big(v_1, ..., v_k\big) \text{ and } W = \big(w_{k+1}, ..., w_n\big) \tag{38}$$

Furthermore, $V^TW = 0_{k\times(n-k)}$ and $W^TV = 0_{(n-k)\times k}$ (39). Then the transformed matrix $\bar{A}$ will be:

$$\bar{A} = TAT^{-1} = S^TAS = \begin{pmatrix} V^T \\ W^T \end{pmatrix} A \begin{pmatrix} V & W \end{pmatrix} = \begin{pmatrix} V^TAV & V^TAW \\ W^TAV & W^TAW \end{pmatrix}. \tag{40}$$

The columns of V are elements of X_c, therefore, the columns of AV are also elements of X_c. The columns of W are the basis vectors of $X_{\bar{c}} = X_c^\perp$, therefore, $W^TAV = 0_{(n-k)\times k}$. The transformed matrix $\bar{B}$ will be:

$$\bar{B} = TB = S^TB = \begin{pmatrix} V^T \\ W^T \end{pmatrix} B = \begin{pmatrix} V^TB \\ W^TB \end{pmatrix}. \tag{41}$$

Since $B \in X_c$, $w_j \in X_c^\perp$, $W^TB = 0_{(n-k)\times 1}$, $j = \overline{k+1, n}$. □

B.4 Observability staircase form

Proposition 16. Observability staircase form

We construct the following transformation matrix $T^{-1} = S = \big(v_1, ..., v_k, w_{k+1}, ..., w_n\big)$, where $[v] = [v_1, ..., v_k]$ is the orthonormal (ON) basis of $X_o = \mathrm{Ker}(\mathcal{O}_n)^\perp = \mathrm{Im}\big(\mathcal{O}_n^T\big)$ and $[w] = [w_{k+1}, ..., w_n]$ is the ON basis of $X_{\bar{o}} = \mathrm{Ker}(\mathcal{O}_n)$. Then the transformed matrices will have the form:

$$\bar{A} = TAT^{-1} = \begin{pmatrix} \bar{A}_{11} & 0_{k\times(n-k)} \\ \bar{A}_{21} & \bar{A}_{22} \end{pmatrix} \tag{42a}$$

$$\bar{C} = CT^{-1} = \begin{pmatrix} \bar{C}_1 & 0_{1\times(n-k)} \end{pmatrix} \tag{42b}$$

Using SVD: $\mathcal{O}_n = U_o\Sigma_oV_o^T$, $S := V_o$

Proof. (For simplicity, only for SISO) Since X_o and $X_{\bar{o}}$ are orthogonal complement of each other (i.e. $X_o \otimes X_{\bar{o}} = \mathbb{R}^n$), $[v, w]$ is an ON basis of $\mathbb{R}^n$. In other words: S is an orthogonal matrix with the well-known properties:

$$S^T S = I_n \quad \Rightarrow \quad S^{-1} = S^T = \begin{pmatrix} V^T \\ W^T \end{pmatrix}, \quad \text{where } V = \left(v_1, ..., v_k\right) \text{ and } W = \left(w_{k+1}, ..., w_n\right) \tag{43}$$

Furthermore, $V^T W = 0_{k\times(n-k)}$ and $W^T V = 0_{(n-k)\times k}$ (44). The transformed matrix $\bar{A}$ will be:

$$\bar{A} = TAT^{-1} = S^T AS = \begin{pmatrix} V^T \\ W^T \end{pmatrix} A \begin{pmatrix} V & W \end{pmatrix} = \begin{pmatrix} V^T AV & V^T AW \\ W^T AV & W^T AW \end{pmatrix}. \tag{45}$$

The columns of V are elements of X_o, therefore, the columns of $A^T V$ are also elements of X_o. The columns of W are the basis vectors of $X_{\bar{c}} = X_c^\perp$, therefore, $(A^T V)^T W = V^T AW = 0_{k\times(n-k)}$. The transformed matrix $\bar{C}$ will be:

$$\bar{C} = CT^{-1} = CS = C \begin{pmatrix} V & W \end{pmatrix} = \begin{pmatrix} CV & CW \end{pmatrix}. \tag{46}$$

Since $C^T \in X_o$, $w_j \in X_o^\perp$, $CW^T = 0_{1\times(n-k)}$, $j = \overline{k+1, n}$. □

Proposition 17. If (A, C) has unobservable mode (i.e. is unobservable), there exists $x \in \mathbb{R}^n$, such that $Ax = \lambda x$ and $Cx = 0$. Consequently, λ is a "decoupling zero" of (A, B, C, D), since

$$M = \begin{pmatrix} A - \lambda I & B \\ C & 0 \end{pmatrix} \text{ is singular,} \tag{47}$$

namely there exists $\xi = \begin{pmatrix} x \\ 0 \end{pmatrix} \neq 0$ such that $M\xi = 0$. Or in other words, the kernel space of M is not empty, meaning that M is singular.

Proposition 18. The input decoupling zeros are equal to the eigenvalues of the uncontrollable subsystem.

Proof. We assume that (A, B) is uncontrollable:

$$\mathcal{C}_n = \begin{pmatrix} B & AB & ... & A^{n-1}B \end{pmatrix} \in \mathbb{R}^{n\times mn} \tag{48}$$

is rank deficient, that implies a nonempty kernel space $\text{Ker}\left(\mathcal{C}_n^T\right) \subset \mathbb{R}^n$, namely, there exists $x \in \mathbb{R}^n$ such that $\mathcal{C}_n^T x = 0$. Alternatively, we have that

$$\begin{cases} B^T x = 0 \\ B^T A^T x = 0 \\ \cdots \\ B^T \left(A^T\right)^{n-1} x = 0 \end{cases} \tag{49}$$

□

B.5 Kalman decomposition

We produce a controllability staircase form decomposition on the system, than on both subsystems (controllable and uncontrollable) we produce an observability staircase form decomposition.