# Computer controlled systems

Lecture 3

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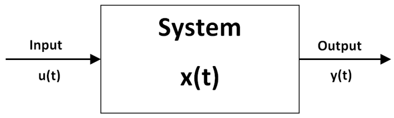
## 1 System representations



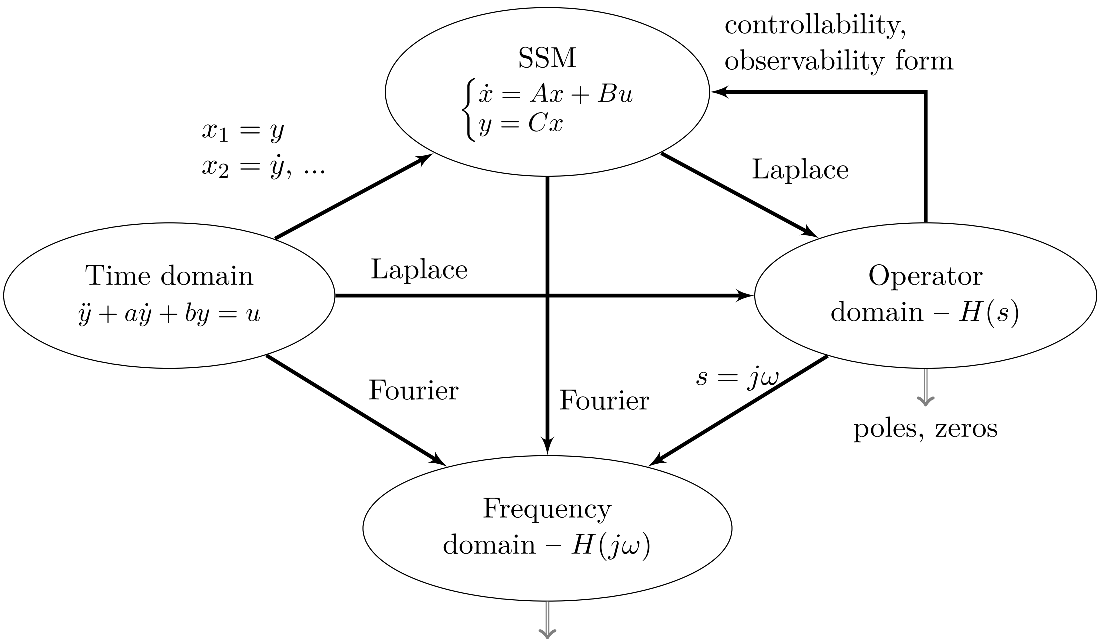




Figure 1. System representations, SSM (State Space Model) – ÁTM (Állapottér model)

## 1.1 Time domain → Operator domain

**Example 1.** The system's differential equation:

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = 2\ddot{u}(t) - 3\dot{u}(t) + u(t)$$

Assume that the initial conditions are all zero. The Laplace transform of the above equation is:

$$s^2Y(s) + 3sY(s) + 2Y(s) = 2s^2U(s) - 3sU(s) + U(s)$$

$$(s^2 + 3s + 2)Y(s) = (2s^2 - 3s + 1)U(s)$$

From this, we obtain the system's transfer function:

$$H(s) = \frac{Y(s)}{U(s)} = \frac{2s^2 - 3s + 1}{s^2 + 3s + 2} = \underbrace{\frac{-9s - 3}{s^2 + 3s + 2}}_{C(sI-A)^{-1}B} + \underbrace{2}_{D} \quad \Rightarrow$$

Controller, observer form (in advance)

$$\begin{cases} \dot{x} = \begin{pmatrix} -3 & -2 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u \\ y = \begin{pmatrix} -9 & -3 \end{pmatrix} x + 2u \end{cases} \quad \text{ctr. form}$$

$$\begin{cases} \dot{x} = \begin{pmatrix} -3 & 1 \\ -2 & 0 \end{pmatrix} x + \begin{pmatrix} -9 \\ -3 \end{pmatrix} u \\ y = \begin{pmatrix} 1 & 0 \end{pmatrix} x + 2u \end{cases} \quad \text{obs. form}$$

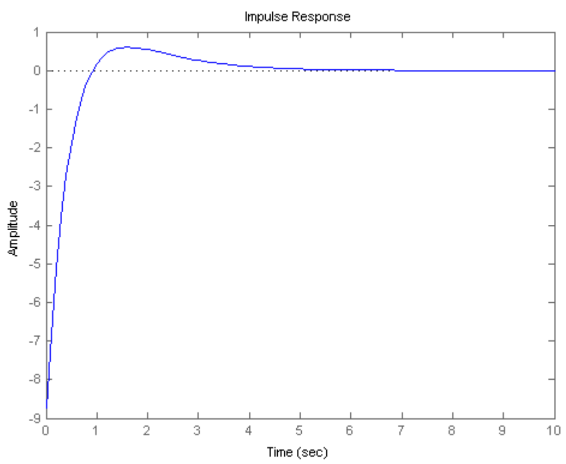
From $H(s)$, we can determine the system's impulse response function (using inverse Laplace transformation):

$$H(s) = 2 + \frac{-9s - 3}{(s+1)(s+2)} = 2 + \frac{C_1}{s+1} + \frac{C_2}{s+2}$$

$$C_i = \lim_{s \to \alpha_i} (s - \alpha_i)H(s)$$

$$C_1 = 6 \quad C_2 = -15$$

$$h(t) = \mathfrak{L}^{-1}\{H(s)\} = \mathfrak{L}^{-1}\{2 + \frac{6}{s+1} + \frac{-15}{s+2}\} = 2\delta(t) + 6e^{-t} - 15e^{-2t}$$




In a strictly proper system the input does not affect the output *directly*. In the operator domain, this means that the degree of the transfer function's numerator is less then the degree of the denominator. In the state space model, matrix $D$ is zero if the system is strictly proper.

Figure 2. Impulse response $h(t)$ of the system.

**Matlab 1.** conv,deconv — Polynomial multiplication and division

Polynome multiplication

$$(s^2 + 3s + 2) \cdot (s + 4) = s^3 + 7s^2 + 14s + 8$$

```
>> C = conv([1 3 2],[1 4])
C = 1      7      14      8
```

Polynome division

$$\frac{2s^2-3s+1}{s^2+3s+2} = \frac{-9s-3}{s^2+3s+2} + 2$$

```
>> [Q,R] = deconv([2 -3 1],[1 3 2])
Q = 2
R = 0     -9     -3
```

**Matlab 2.** ss2tf — Compute the transfer function of a SSM

```
>> [num,den] = ss2tf([-3 -2 ; 1 0],[1 0]',[-9 -3],2)
num = 2     -3      1
den = 1      3      2
```

## 1.2 Time domain → SSM

State space model: *system* of first order linear differential equations (elsrend differenciálegyenletekbl álló egyenletrendszer)

$$\dot{x} = f(x) + g(x)u, \quad \text{where } f : \mathbb{R}^n \to \mathbb{R}^n, \quad g : \mathbb{R}^n \to \mathbb{R}^{n\times r}, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^r. \tag{1}$$

Linear case:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{2}$$

**Example 2.** It is given the following second order linear scalar differential equation

$$\ddot{y} = -y$$

We introduce the following notation:

$$x_1 = y \ , \quad x_2 = \dot{y} = \dot{x}_1$$

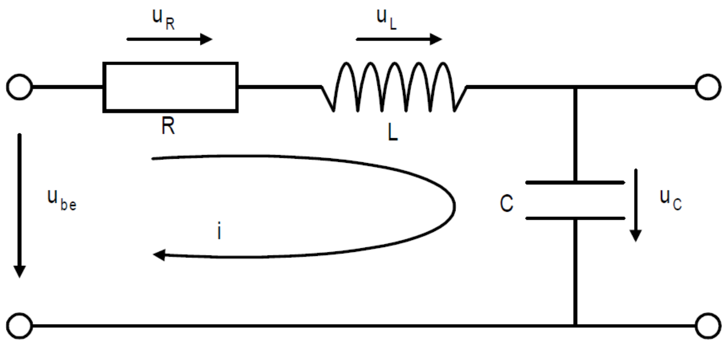
SSM:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \ddot{x}_1 = \ddot{y} = -y = -x_1 \end{aligned}$$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

Usually, the procedure for (time domain → SSM) is the following:
Linear higher order scalar differential equation with constant coefficient → compute its transfer function $H(s)$ → SSM (Controller or Observer form).

**Example 3.**




The system's differential equation ($u_{be}$ stands for $u_{in}$ – is the input voltage):

$$u_{be} = u_R + u_L + u_C$$

$$i = C \cdot \frac{\mathrm{d}u_C}{\mathrm{d}t}, \quad u_L = L \cdot \frac{\mathrm{d}i}{\mathrm{d}t}, \quad u_R = R \cdot i$$

State equations (let the state variables be: $i$, $u_C$):

$$\begin{cases} u_{be} = R \cdot i + L \cdot \dfrac{\mathrm{d}i}{\mathrm{d}t} + u_C \\ i = C \cdot \dfrac{\mathrm{d}u_C}{\mathrm{d}t} \end{cases} \quad \Rightarrow \quad \begin{cases} \dfrac{\mathrm{d}i}{\mathrm{d}t} = -\dfrac{R}{L}i - \dfrac{1}{L}u_C + \dfrac{1}{L}u_{be} \\ \dfrac{\mathrm{d}u_C}{\mathrm{d}t} = \dfrac{1}{C}i \end{cases} \tag{3}$$

In matrix form:

$$\begin{pmatrix} \frac{\mathrm{d}i}{\mathrm{d}t} \\ \frac{\mathrm{d}u_C}{\mathrm{d}t} \end{pmatrix} = \begin{pmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{pmatrix} \cdot \begin{pmatrix} i \\ u_C \end{pmatrix} + \begin{pmatrix} \frac{1}{L} \\ 0 \end{pmatrix} u_{be}$$

Hence, the state vector is $x(t) = \begin{pmatrix} i \\ u_C \end{pmatrix}$, $u(t) = u_{be}$

We must define the output of the system: $y(t) = u_C = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} i \\ u_C \end{pmatrix}$

Now we consider a concrete numerical example. Let $R = 1.5\Omega$, $L = 0.25H$, $C = 0.5F$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -6 & -4 \\ 2 & 0 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 4 \\ 0 \end{pmatrix} u$$

$$y = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$A = \begin{pmatrix} -6 & -4 \\ 2 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \quad C = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

In Figure 3, you can see the impulse and step response of the system. Furthermore, Figure 4 illustrates the system's output in case of a sinusoidal input function $u(t) = 2\sin(3t)$ (szinuszos gerjesztés).

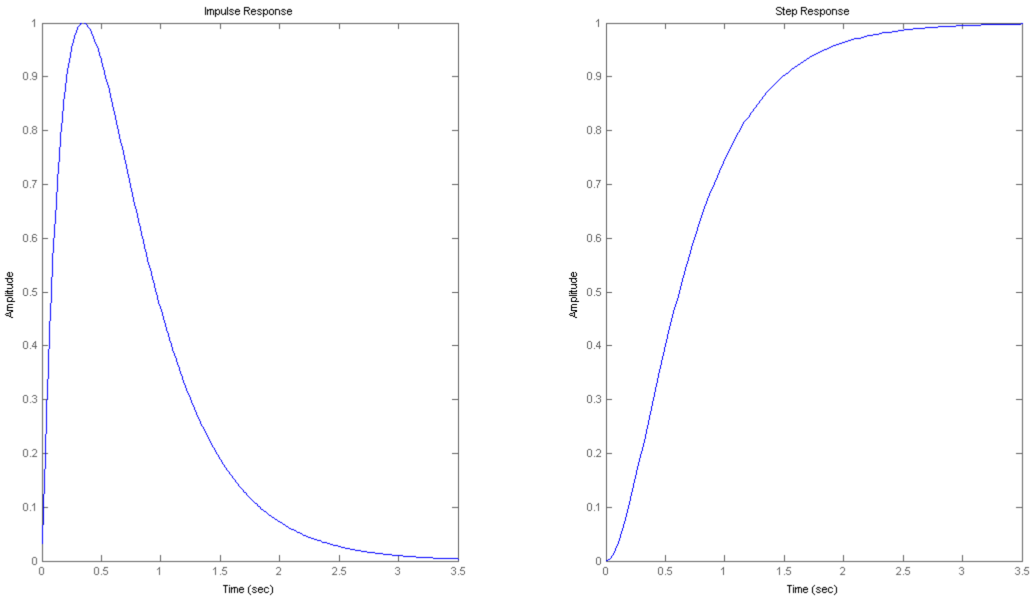



Figure 3. Impulse and step response of the RLC circuit

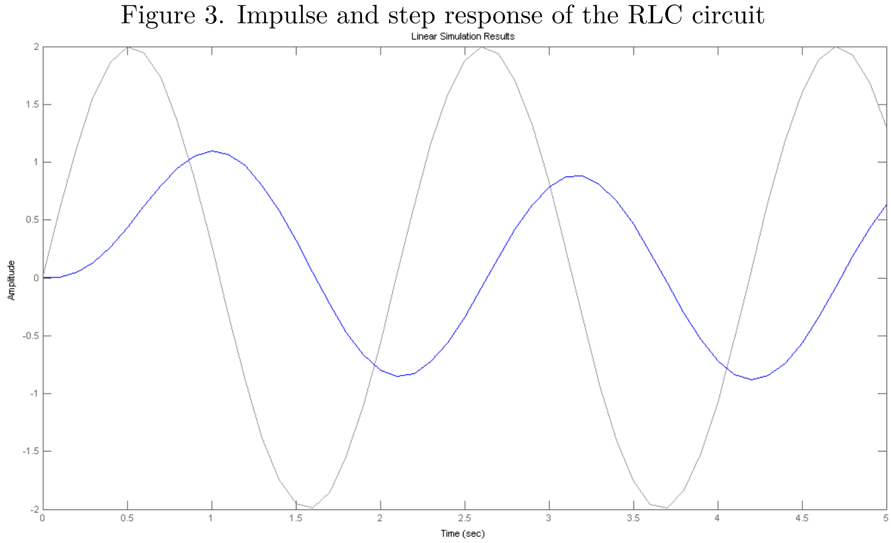



Figure 4. Excitation (gerjesztés) of the RLC circuit with a sinusoidal input function. In case of $f = 3$Hz frequency signal the system's gain is $g = 0.4438$, phase shift is $\phi = -93.1798°$. HU: 3 Hz esetén a rendszer egyenáramú ersítése $g = 0.4438$, fáziseltolása $\phi = -93.1798°$.

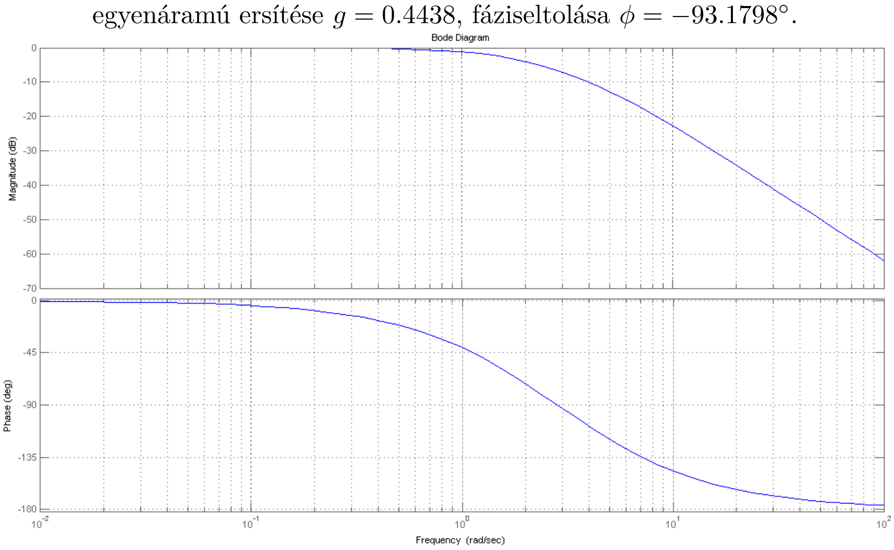



Figure 5. Bode diagram of the system describing the RLC circuit.

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**Example 4.** (Diagonal SSM)

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -6 & -4 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 4 \\ 0 \end{pmatrix} u$$

$$y = \begin{pmatrix} 0 & 1 \end{pmatrix} x$$

We compute the eigenvalue/eigenvector decomposition (alternatively: ***spectral decomposition***) of matrix A

$$\begin{vmatrix} \lambda + 6 & 4 \\ -2 & \lambda \end{vmatrix} = (\lambda + 6)\lambda + 8 = (\lambda + 2)(\lambda + 4)$$

Eigenvalues: $\lambda_1 = -2$, $\lambda_2 = -4$

Eigenvectors in case of $\lambda_1 = -2$:

$$\begin{pmatrix} -6 & -4 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = -2 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \rightarrow \begin{matrix} -6v_1 - 4v_2 = -2v_1 \rightarrow v_2 = -v_1 \\ 2v_1 = -2v_2 \end{matrix}$$

$v = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot p \qquad p \in \mathbb{R} \setminus \{0\}$

Eigenvectors in case of $\lambda_2 = -4$:

$$\begin{pmatrix} -6 & -4 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = -4 \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \rightarrow \begin{matrix} -6w_1 - 4w_2 = -4w_1 \rightarrow w_1 = -2w_2 \\ 2w_1 = -4w_2 \end{matrix}$$

$w = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \cdot q \qquad q \in \mathbb{R} \setminus \{0\}$

Transformation matrix:

$$T = S^{-1} \quad T^{-1} = S$$

$$T^{-1} = \begin{pmatrix} v & w \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix} \rightarrow T = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix}$$

State space transformation:

$$\bar{A} = TAT^{-1} = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} -6 & -4 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & -4 \end{pmatrix}$$

$$\bar{B} = TB = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 4 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ -4 \end{pmatrix}$$

$$\bar{C} = CT^{-1} = \begin{pmatrix} -1 & 1 \end{pmatrix}$$

## 1.3 SSM → Operator domain

$$\begin{matrix} \dot{x} = Ax + Bu \\ y = Cx + Du \\ x(0) = 0 \end{matrix} \rightarrow \begin{matrix} sIX(s) = AX(s) + BU(s) \rightarrow X(s) = (sI - A)^{-1}BU(s) \\ Y(s) = CX(s) + DU(s) = (C(sI - A)^{-1}B + D)U(s) \end{matrix}$$

$$H(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D$$

**Example 5.**

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -6 & -4 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 4 \\ 0 \end{pmatrix} u$$

$$y = \begin{pmatrix} 0 & 1 \end{pmatrix} x$$

$$H(s) = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} s+6 & 4 \\ -2 & s \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 1 \end{pmatrix} \frac{1}{(s+6)s+8} \begin{pmatrix} s & -4 \\ 2 & s+6 \end{pmatrix} \begin{pmatrix} 4 \\ 0 \end{pmatrix} = \frac{1}{s^2+6s+8} \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 4s \\ 8 \end{pmatrix} = \frac{8}{(s+2)(s+4)}$$

**Proposition 1.** (special case – if the SSM is diagonal, i.e. matrix $A$ is diagonal)

$$H(s) = C(sI - A)^{-1}B = \begin{pmatrix} c_1 \cdots c_n \end{pmatrix} \begin{pmatrix} s-\lambda_1 & 0 & \cdots & 0 \\ 0 & s-\lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & s-\lambda_n \end{pmatrix}^{-1} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} =$$

$$= \begin{pmatrix} c_1 \cdots c_n \end{pmatrix} \begin{pmatrix} \frac{1}{s-\lambda_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{s-\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{s-\lambda_n} \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \frac{c_1 b_1}{s-\lambda_1} + \frac{c_2 b_2}{s-\lambda_2} + \cdots + \frac{c_n b_n}{s-\lambda_n} = \sum_{i=1}^{n} \frac{c_i b_i}{s-\lambda_i}$$

**Example 6.**

$$\dot{x} = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -6 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \\ -10 \\ 2 \end{pmatrix} u$$

$$y = \begin{pmatrix} 6 & 8 & 2 & -1 \end{pmatrix} x$$

$$H(s) = \frac{0 \cdot 6}{s-4} + \frac{1 \cdot 8}{s+3} + \frac{(-10) \cdot 2}{s+2} + \frac{2 \cdot (-1)}{s+6} = \frac{8}{s+3} - \frac{20}{s+2} - \frac{2}{s+6}$$

## 1.4 Operator domain → SSM : Controller form

**Theorem 2.** (Controller, observer form – only SISO)

$$H(s) = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_0}{1s^n + a_{n-1}s^{n-1} + \cdots + a_0} + D = \frac{b(s)}{a(s)} + D\,, \qquad D \in \mathbb{R}$$

The denominator must always be monic (with leading coefficient 1).
A nevezben szerepl polinom vezéregyütthatója mindig 1 kell, hogy legyen!

$$\text{controller form:} \quad \begin{cases} \dot{x} = \begin{pmatrix} -a_{n-1} & -a_{n-2} & \cdots & -a_1 & -a_0 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} u \\ y = \begin{pmatrix} b_{n-1} & b_{n-2} & \cdots & b_0 \end{pmatrix} x + Du \end{cases} \tag{4}$$

$$\text{observer form:} \quad \begin{cases} \dot{x} = \begin{pmatrix} -a_{n-1} & 1 & \cdots & 0 & 0 \\ -a_{n-2} & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ -a_1 & 0 & \cdots & 1 & 0 \\ -a_0 & 0 & \cdots & 0 & 0 \end{pmatrix} x + \begin{pmatrix} b_{n-1} \\ b_{n-2} \\ \vdots \\ b_0 \end{pmatrix} u \\ y = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix} x + Du \end{cases} \tag{5}$$

**Example 7.**

$$H(s) = \frac{4s + 38}{(s+1)(s+2)(2s+6)} = \frac{4s + 38}{2s^3 + 12s^2 + 22s + 12} = \frac{2s + 19}{s^3 + 6s^2 + 11s + 6}$$

$$\dot{x} = \begin{pmatrix} -6 & -11 & -6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} u$$

$$y = \begin{pmatrix} 0 & 2 & 19 \end{pmatrix} x$$

*Remark.* The controller form produces a controllable SSM!
A controller form mindig irányítható ÁTM-et eredményez! (Lásd később)

Check whether it leads, indeed, to the initial transfer function:

$$H(s) = C(sI - A)^{-1}B = \begin{pmatrix} 0 & 2 & 19 \end{pmatrix} \begin{pmatrix} s+6 & 11 & 6 \\ -1 & s & 0 \\ 0 & -1 & s \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 2 & 19 \end{pmatrix} \frac{1}{s^2(s+6) + 6 + 11s} \begin{pmatrix} s^2 & -11s - 6 & 6s \\ s & s(s+6) & -6 \\ 1 & s+6 & s(s+6) + 11 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} =$$

$$\frac{1}{s^3 + 17s + 6} \begin{pmatrix} 0 & 2 & 19 \end{pmatrix} \begin{pmatrix} s^2 \\ s \\ 1 \end{pmatrix} = \frac{2s + 19}{s^3 + 6s^2 + 11s + 6}$$

! **Only SISO case.** The controller and observer forms are very similar:

$$A_c = A_o^T, \quad B_c = C_o^T, \quad C_c = B_o^T \tag{6}$$

SISO $\Rightarrow C(sI - A)^{-1}B \in \mathbb{R}$, thus

$$C(sI - A)^{-1}B = \left(C(sI - A)^{-1}B\right)^T = B^T(sI - A)^{-T}C^T = B^T(sI - A^T)^{-1}C^T \tag{7}$$