Computer Controlled Systems
Lecture 8

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Outline

Short revision

Remember:

- characteristic polynomial of square matrix A: $a(s) = \det(sI - A)$
- asymptotic stability of LTI state space model (A, B, C, D): each eigenvalue of A have negative real part
- roots of $a(s) \equiv$ eigenvalues of $A \equiv$ poles of the transfer function $H(s) = C(sI - A)^{-1}B + D$

Motivation:

- we would like to change the poles of a LTI system by appropriate feedback (e.g., the original system is unstable and we want to stabilize it, or we want to improve performance, eliminate oscillations etc.)

General problem statement

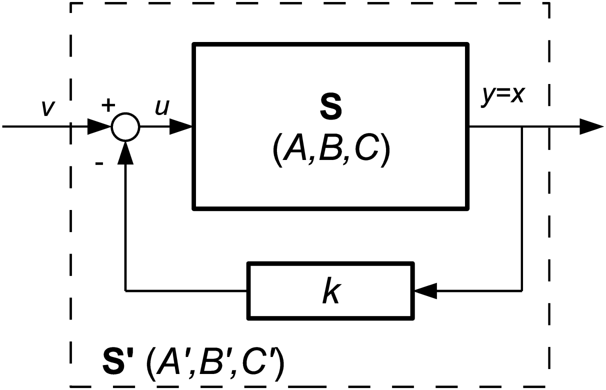
Given:

- a ***SISO LTI*** system with matrices (A, B, C).
  The poles depend on A (on $a(s)$).
- prescribed (expected) poles defined by polynomial $\alpha(s)$, such that
  $\deg\ a(s) = \deg\ \alpha(s) = n$

To be computed:
a ***full state feedback*** such that the poles of the closed loop system will be the roots of $\alpha(s)$.
Subproblem: feedback design, which can stabilize an otherwise unstable system.

Closed loop LTI system – 1

Static linear (full) state feedback:



$$u = -kx + v,$$

where $k \in \mathbb{R}^{r\times n}$, ha $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^r$

Closed LTI system – 2

The matrices of the SISO system are (A, B, C)

$$\begin{aligned}
&\dot{x}(t) = Ax(t) + Bu(t) \\
&y(t) = Cx(t) \\
&y(t)\ ,\ u(t) \in \mathbb{R} \quad , \quad x(t) \in \mathbb{R}^n \\
&A \in \mathbb{R}^{n\times n}\ ,\ B \in \mathbb{R}^{n\times 1}\ ,\ C \in \mathbb{R}^{1\times n}
\end{aligned}$$

static linear full state feedback

$$\begin{aligned}
&v = u + kx \quad (u = v - kx) \\
&k = \begin{bmatrix} k_1 & k_2 & . & . & . & k_n \end{bmatrix} \\
&k \in \mathbb{R}^{1\times n} \qquad \text{(row vector)}
\end{aligned}$$

Closed LTI system – 3

Closed loop system

$$\dot{x}(t) = (A - Bk)x(t) + Bv(t)$$
$$y(t) = Cx(t)$$

Namely:

$$A' = A - B \cdot k, \quad B' = B, \quad C' = C$$

Characteristic polynomials
Without feedback (uncontrolled system):

$$a(s) = \det(sI - A)$$

Closed loop system (controlled) system:

$$a_c(s) = \det(sI - A + Bk)$$

Determinant of block matrices

Let us calculate the following determinant

$$\det \begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix}$$

in two different (but equivalent) ways

$$\det(M_1)\det(M_4 - M_3 M_1^{-1} M_2) = \det(M_4)\det(M_1 - M_2 M_4^{-1} M_3)$$

Let's construct the following matrix:

$$\det \begin{bmatrix} sI - A & B \\ -k & 1 \end{bmatrix}$$

then we obtain the following:

$$\det(sI - A)\det(1 + k(sI - A)^{-1}B) = 1 \cdot \det((sI - A) + B \cdot 1^{-1} \cdot k)$$

Resolvent formula

$$a(s) = s^n + a_1 s^{n-1} + \dots + a_n$$
$$(sI - A)^{-1} = \frac{1}{a(s)}(s^{n-1}I + s^{n-2}(A + a_1 I) + s^{n-3}(A^2 + a_1 A + a_2 I) + \dots)$$

Proof:

$$(sI - A)(sI - A)^{-1} =$$
$$(sI - A)\frac{1}{a(s)}(s^{n-1}I + s^{n-2}(A + a_1 I) + s^{n-3}(A^2 + a_1 A + a_2 I) + \dots) =$$
$$= \frac{1}{a(s)}\left[s^n I \underbrace{-s^{n-1}A + s^{n-1}A}_{0} + a_1 s^{n-1} I - s^{n-2}A^2 - s^{n-2}a_1 A + \dots\right] =$$
$$\frac{a(s)}{a(s)} I = I$$

Pole placement – 1

$$det(sI - A) \cdot det(1 + k(sI - A)^{-1}B) = 1 \cdot det((sI - A) + B \cdot 1^{-1} \cdot k)$$

$$a(s)(1 + k(sI - A)^{-1}B) = det(sI - A + Bk)$$

$$\alpha(s) = a(s)(1 + k(sI - A)^{-1}B) \quad \Rightarrow \quad \alpha(s) - a(s) = a(s)k(sI - A)^{-1}B$$

Using the ***resolvent formula***

$$(sI - A)^{-1} = \frac{1}{a(s)}(s^{n-1}I + s^{n-2}(A + a_1 I) + s^{n-3}(A^2 + a_1 A + a_2 I) + ...$$

we obtain that

$$\begin{aligned}(\alpha_1 - a_1)s^{n-1} + (\alpha_2 - a_2)s^{n-2} + ...(\alpha_n - a_n) = \\ = kBs^{n-1} + k(A + a_1 I)Bs^{n-2} + ...\end{aligned}$$

Pole placement – 2

$$(\alpha_1 - a_1)s^{n-1} + (\alpha_2 - a_2)s^{n-2} + \ldots(\alpha_n - a_n) = kBs^{n-1} + k(A + a_1 I)Bs^{n-2} + \ldots$$

polynomial equation

$$\begin{aligned}
\alpha_1 - a_1 &= kB \\
\alpha_2 - a_2 &= kAB + a_1 kB = a_1 kB + kAB \\
\alpha_3 - a_3 &= kA^2B + a_1 kAB + a_2 kB = a_2 kB + a_1 kAB + kA^2B \\
&. \\
&.
\end{aligned}$$

$$\underline{\alpha} - \underline{a} = k \begin{bmatrix} B & AB & A^2B & \ldots & A^{n-1}B \end{bmatrix} \begin{bmatrix} 1 & a_1 & a_2 & . & . & . & a_{n-1} \\ 0 & 1 & a_1 & . & . & . & a_{n-2} \\ 0 & 0 & 1 & . & . & . & a_{n-3} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{bmatrix}$$

Pole placement controller

$$\underline{\alpha} - \underline{a} = k \begin{bmatrix} B & AB & A^2B & \ldots & A^{n-1}B \end{bmatrix} \begin{bmatrix} 1 & a_1 & a_2 & . & . & . & a_{n-1} \\ 0 & 1 & a_1 & . & . & . & a_{n-2} \\ 0 & 0 & 1 & . & . & . & a_{n-3} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{bmatrix}$$

$$\underline{\alpha} - \underline{a} = k\mathcal{C}T_\ell^T$$

If **S** is ***controllable*** then

$$k = (\underline{\alpha} - \underline{a})T_\ell^{-T}\mathcal{C}^{-1}$$

Controller form realization

$$\dot{x}(t) = A_c x(t) + B_c u(t)$$
$$y(t) = C_c x(t)$$

where

$$A_c = \begin{bmatrix} -a_1 & -a_2 & . & . & . & -a_n \\ 1 & 0 & . & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & 1 & 0 \end{bmatrix}, \quad B_c = \begin{bmatrix} 1 \\ 0 \\ . \\ . \\ . \\ 0 \end{bmatrix}$$

$$C_c = \begin{bmatrix} b_1 & b_2 & . & . & . & b_n \end{bmatrix}$$

The polynomials of the transfer function
$a(s) = s^n + a_1 s^{n-1} + ... + a_{n-1} s + a_n$ and $b(s) = b_1 s^{n-1} + ... + b_{n-1} s + b_n$
$H(s) = \frac{b(s)}{a(s)}$

Pole placement controller in case of a controller form

$$A_c - B_c k_c = \begin{bmatrix} -(a_1 + k_{c1}) & -(a_2 + k_{c2}) & . & . & . & -(a_n + k_{cn}) \\ 1 & 0 & . & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & 1 & 0 \end{bmatrix}$$

the characteristic polynomial of the closed loop system is $\alpha(s)$:

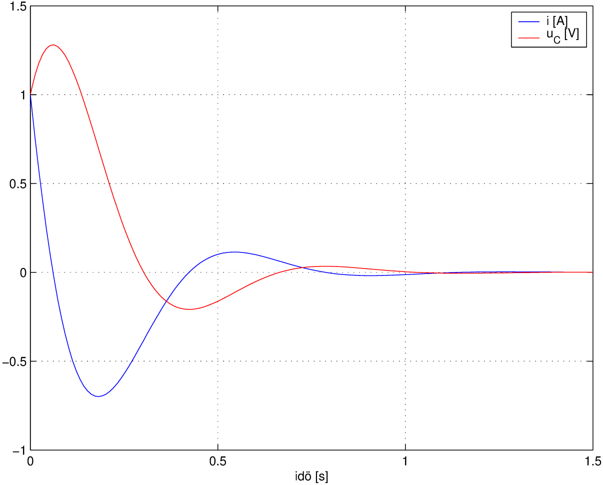
$$\alpha(s) = det(sI - (A_c - B_c k_c)) = s^n + (a_1 + k_{c1})s^{n-1} + ... + (a_n + k_{cn})$$

The coefficients k_c of the state feedback gain is

$$k_c = \underline{\alpha} - \underline{a}$$

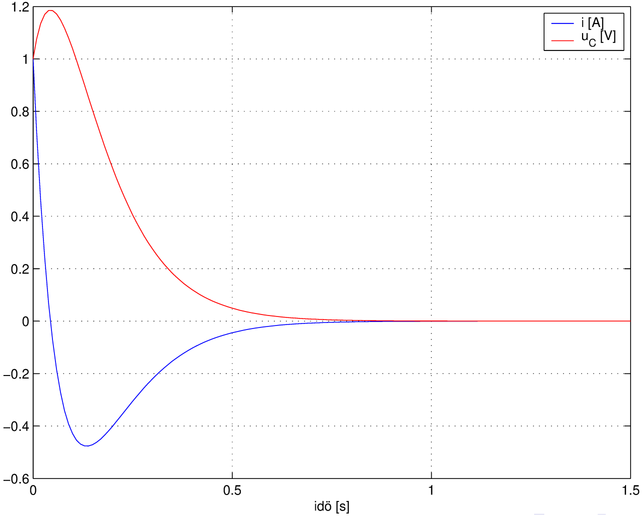
Example – 1

System: RLC circuit. Response of the uncontrolled (open loop) circuit with zero input ($u = 0V$) from initial state $x(0) = [1\ 1]^T$.
(Poles: $-5 \pm 8.6603i$)



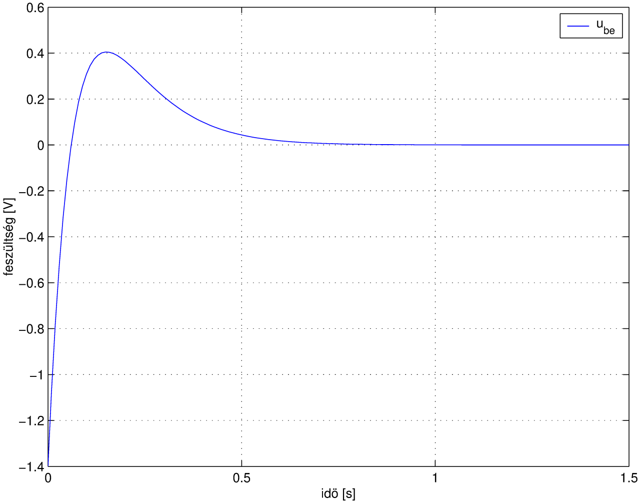
Example – 2

Prescribed poles of the closed loop system: -10, -12. Feedback gain: $k = [1.2 \;\; 0.2]$. Response for $x(0) = [1 \; 1]^T$:



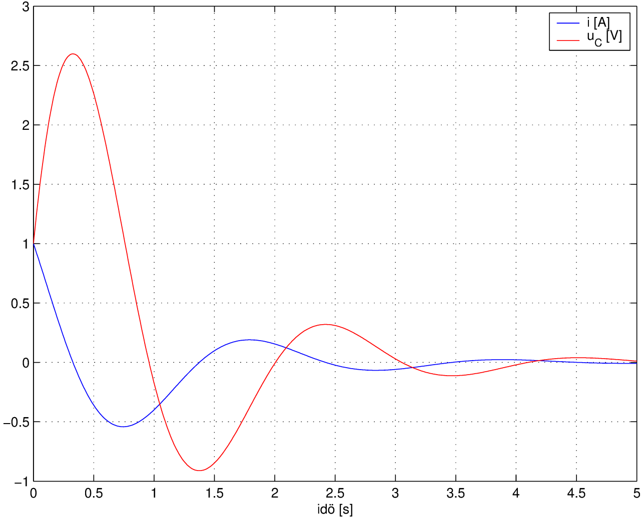
Example – 3

The necessary input for stabilizing control (voltage):



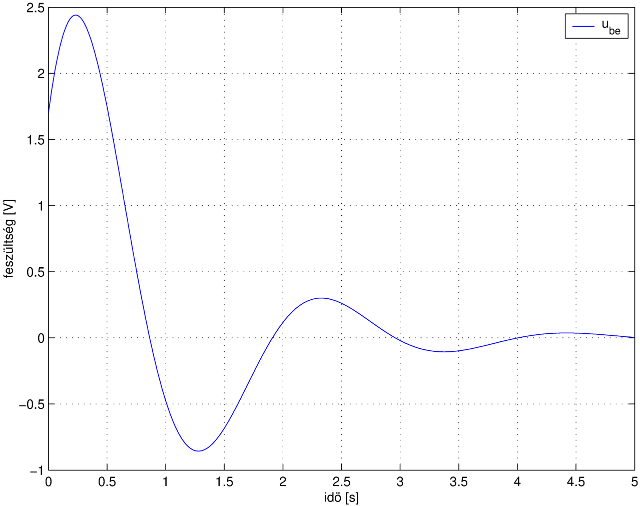
Example – 4

Prescribed poles of the closed loop system: $-1+3i$, $-1-3i$. Feedback gain: $k = [-0.8 \quad -0.9]$. Response:



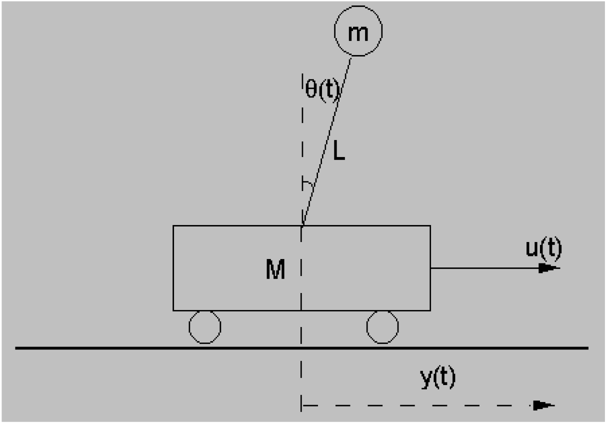
Example – 5

The necessary input for stabilizing control:



Example – 6

System: the inverted pendulum



Example – 7

State vector:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} y \\ \theta \\ \dot{y} \\ \dot{\theta} \end{bmatrix} \quad (1)$$

Equilibrium point: $x^* = [0\ 0\ 0\ 0]^T$
The linearized state-space model:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{mg}{M} & 0 & 0 \\ 0 & \frac{(M+m)g}{ML} & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{M} \\ -\frac{1}{ML} \end{bmatrix}, \quad C = I^{4\times 4}$$

Parameters: $m = 0.5$ kg, $M = 0.1$ kg, $L = 1$ m, $g = 10\ \frac{\text{m}}{\text{s}^2}$

Example – 8

The poles of the uncontrolled system: λ_1=0, λ_2=0, $\lambda_3 = 7.746$, $\lambda_4 = -7.746$

Goal: stabilizing controller

Prescribed poles of the closed loop system: $\kappa_1 = \kappa_2 = \kappa_3 = \kappa_4 = -1$

The computed feedback gain:

$$k = [-0.01 \quad -6.61 \quad -0.04 \quad -0.44]$$

Example – 9

The operation of the controlled system (simulation: Faludi Gábor)

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State observer, problem statement

Recall: If a SSM (A, B, C) is observable, then, knowing the input (u) and the output (y), the initial state of the system can be computed, and hence every further state values.
Problems:

- The measurement of the input and the output are (in general) not precise enough, furthermore, we need the $1\text{st},\ 2\text{nd},\ \ldots,\ (n-1)\text{th}$ derivatives of the output in order to compute the initial condition.
- In general, the system model is not perfect

Goal: design such a tool (state observer), for which we do not need the derivatives of the output y, and the estimated state converges to the actual value of the state vector.

Algebraic form of the state observer

State-space model:

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$
$$\dot{\hat{x}} = (A - LC)\hat{x} + [B \;\; L]\begin{bmatrix} u \\ y \end{bmatrix}$$

observation error:

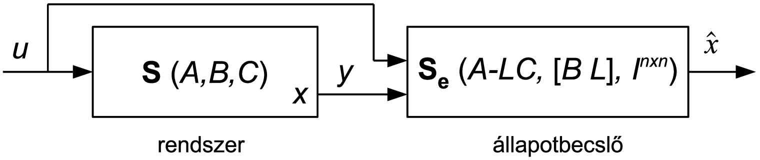
$$e = x - \hat{x}$$

and

$$\dot{e} = (A - LC)e$$

The structure of the state observer

The realization of a state observer (it can be seen from the algebraic equations)



Calculation of the state observer

Reminder: In case of a pole placement controller the system matrices of the closed loop system are $A_c = A - Bk$. (A,B is given, k should be computed, condition: (A,B) is controllable)
System matrix of the state observer: $A_o = A - LC$. (A, C is given, L should be computed, condition: ?)
Solution:

$$A_o^T = A^T - (LC)^T = A^T - C^T L^T$$

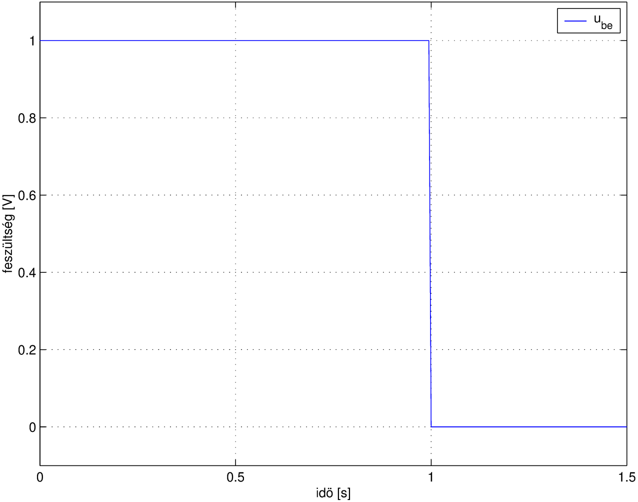
In other words, L can be computed using the pole placement algorithm using arbitrary prescribed stable eigenvalues for A_o (i.e. the state observer be stable). Condition: $[C^T \; A^T C^T \; \ldots \; (A^{n-1})^T C^T] = \mathcal{O}_n^T$ is a full-rank matrix, namely, the system is observable.

Example – 1

RLC circuit, measured output: u_C, namely $C = [0 \;\; 1]$
Prescribed eigenvalues of the state observer: -10, -12
The computed matrix L of the state observer: $L = [-10 \;\; 12]^T$

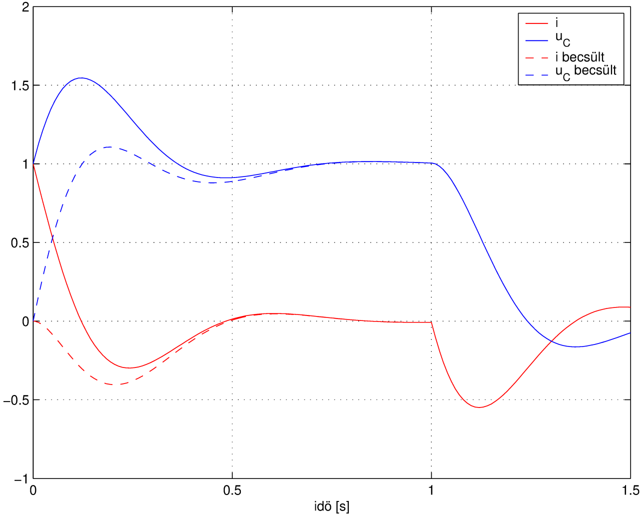
Example – 2

Input of the system:



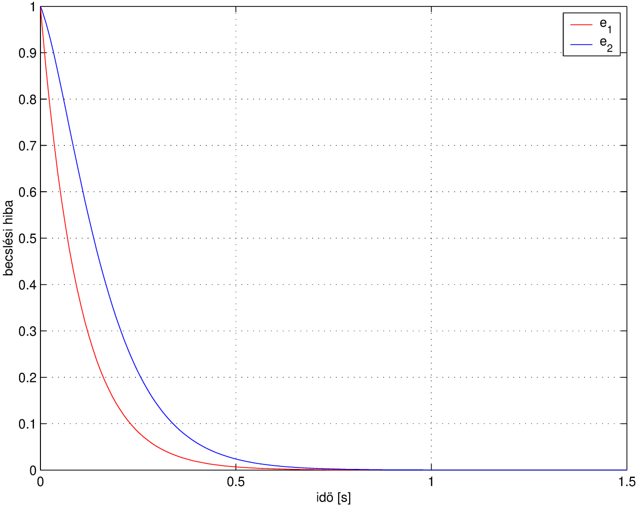
Example – 3

The operation of the state observer:



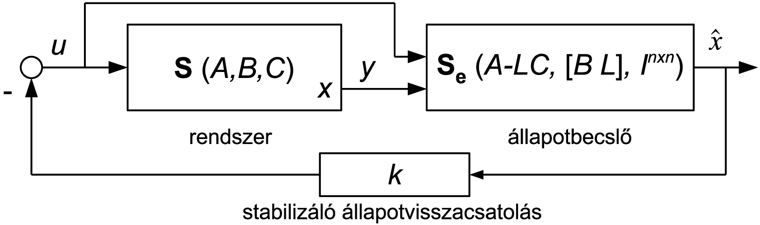
Example – 4

Observation error:



Separation principle

Problem: what happens if the estimated state is fed back by the computed feedback gain k (dynamic output feedback)?



Separation principle: The stabilizing state feedback with the a stable state observer is asymptotically stable, since the dynamics of the closed loop system is the following:

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \cdot \begin{bmatrix} x \\ e \end{bmatrix}$$

This means that the stabilizing state feedback (K) and a stable state observer (L) can be designed separately.

Separation principle

Computation:

$$\dot{x} = Ax + Bu, \;\; u = -K\hat{x}, \;\; \text{and: } e = x - \hat{x}$$

From this: $u = -K(x - e) = -Kx + Ke$, and

$$\dot{x} = Ax + B(-Kx + Ke) = (A - BK)x + BKe \quad (2)$$

$$\dot{e} = (A - LC)e \quad (3)$$

Formula for the eigenvalues:

$$\lambda_i \left(\begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \right) = \lambda_j \left(A - BK \right) \cup \lambda_k \left(A - LC \right),$$

and we know that $A - BK$ ill. $A - LC$ are stability matrices.

Summary

- goal of pole placement: move the poles (eigenvalues) of the controlled system to arbitrary places on the complex plane
- feedback form: full state feedback (requires the knowledge of each state variable)
- condition for computation: controllability
- goal of state observer: asymptotically compute the state variables from the input and the output
- observer gain computation: can be traced back to pole placement (dual problem)
- separation principle: separately designed stabilizing feedback and stable observer results in a stable combined system