

Computer Controlled Systems

Lecture 6

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1st midterm info

- Next week: first midterm
- Time: seminar time for both groups (please check the webpage)
- Possibility for midterm consultation: Next monday between 15:00 and 20:00 (please write me an e-mail about the appropriate starting time)

1 Stability criteria for transfer functions

2 SISO systems in the frequency domain

3 Interconnections of subsystems

Transfer functions and stability

SISO case: $H(s) = C(sI - A)^{-1}B = \frac{b(s)}{a(s)} =$

$$\frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0} = \frac{(s - \beta_1)(s - \beta_2) \dots (s - \beta_m)}{(s - \lambda_1)(s - \lambda_2) \dots (s - \lambda_n)}$$

- Zeros: $\beta_1, \beta_2, \dots, \beta_m \in \mathbb{C}$
- Poles: $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{C}$ (identical to the eigenvalues of A)

Asymptotic stability $\Leftrightarrow \operatorname{Re}(\lambda_i) < 0$

Routh's stability criterion – 1

$$a(s) = a_0s^n + a_1s^{n-1} + \dots + a_{n-1}s + a_n$$

$$\begin{array}{ccccccc} a_0 & & a_2 & & a_4 & & a_6 & \dots \\ a_1 & & a_3 & & a_5 & & a_7 & \dots \\ \frac{a_1a_2 - a_0a_3}{a_1} & & \frac{a_1a_4 - a_0a_5}{a_1} & & \frac{a_1a_6 - a_0a_7}{a_1} & & \dots & \\ \dots & & & & & & & \\ a_n & & \dots & & & & & \end{array}$$

Routh-coefficients: $R_0 = a_0$, $R_1 = a_1$, $R_2 = \frac{a_1a_2 - a_0a_3}{a_1}$, \dots , $R_n = a_n$.
(elements of the first column)

Routh's stability criterion – 2

number of sign changes in the column of coefficients = number of roots with positive real part (unstable)

necessary and sufficient condition for stability: $R_i > 0, i = 0, \dots, n$.

Example: $a(s) = s^3 + s^2 + 3s + 10$.

$R_0 = 1, R_1 = 1, R_2 = -7, R_3 = 10 \Rightarrow 2$ roots with positive real parts (unstable system)

Remarks:

- necessary condition for stability (not sufficient for polynomials with degree greater than 2): all coefficients a_i are positive
- in the case of purely imaginary root(s), zero(s) appear among the coefficients

Hurwitz's stability criterion – 1

$$W = \begin{bmatrix} a_1 & a_3 & a_5 & \dots & 0 & 0 & 0 \\ a_0 & a_2 & a_4 & a_6 & \dots & 0 & 0 \\ 0 & a_1 & a_3 & a_5 & \dots & & 0 \\ 0 & a_0 & a_2 & a_4 & a_6 & \dots & 0 \\ \dots & & & & & & 0 \\ 0 & 0 & 0 & \dots & a_{n-3} & a_{n-1} & 0 \\ 0 & 0 & 0 & \dots & a_{n-4} & a_{n-2} & a_n \end{bmatrix} \in \mathbb{R}^{n \times n}$$

Minors: H_1, H_2, \dots, H_n .

Hurwitz's stability criterion – 2

- necessary and sufficient condition for stability: $H_i > 0, i = 1, \dots, n$
- 0 minor: imaginary root
- negative minor: root with positive real part
- relation between Routh- and Hurwitz-coefficients: $R_i = \frac{H_i}{H_{i-1}}, H_0 = 1.$

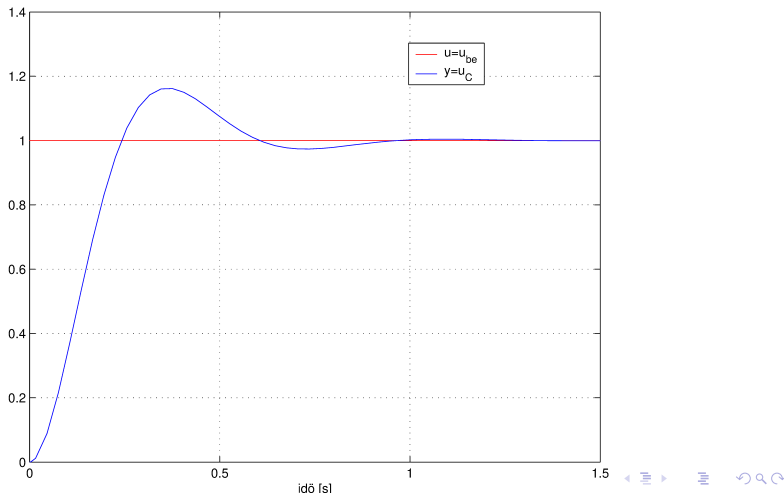
- 1 Stability criteria for transfer functions
- 2 SISO systems in the frequency domain**
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Example: RLC circuit – 1

$$R = 1\Omega, L = 0.1H, C = 0.1F, x(0) = [0 \ 0]^T, y = u_C,$$

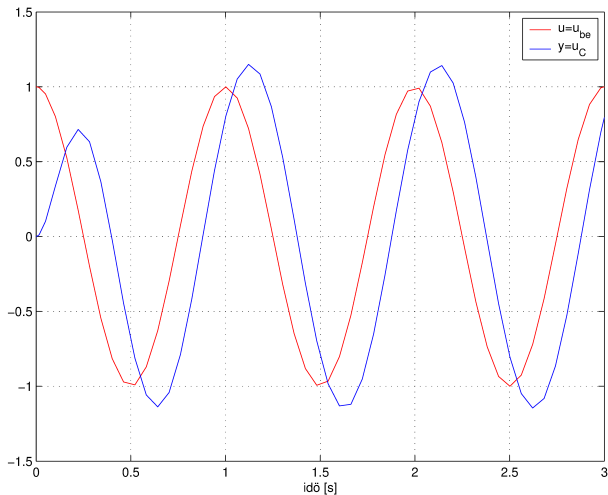
$$u(t) = u_{be} = \cos(\omega \cdot t)$$

$$\omega = 0 \text{ rad/s} = 0 \text{ Hz}$$



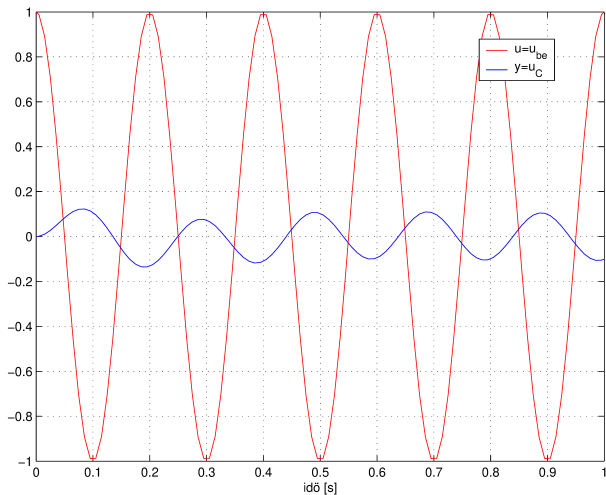
Example: RLC circuit – 2

$$\omega = 2\pi \text{ rad/s} = 1 \text{ Hz}$$



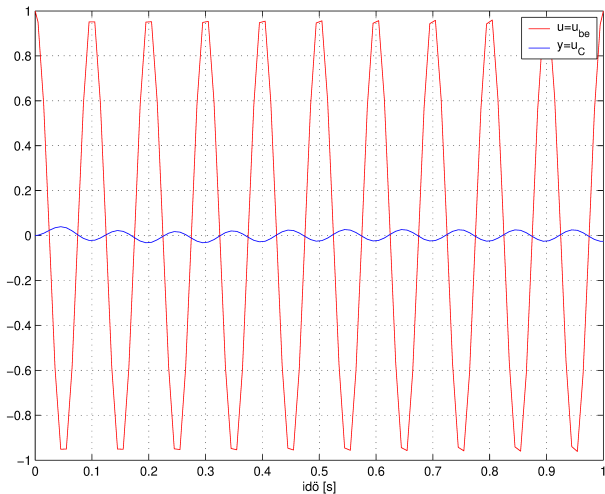
Example: RLC circuit – 3

$$\omega = 5 \cdot 2\pi \text{ rad/s} = 5 \text{ Hz}$$



Example: RLC circuit – 4

$$\omega = 10 \cdot 2\pi \text{ rad/s} = 10 \text{ Hz}$$



Fourier- and Laplace-transforms

Revision: $f : \mathbb{R}_0^+ \mapsto \mathbb{R}$

Fourier-transform:

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t} dt, \quad \omega \in \mathbb{R}$$

Laplace-transform:

$$F(s) = \int_0^{\infty} f(t)e^{-st} dt, \quad s \in \mathbb{C}$$

Assume that s is on the imaginary axis. Then: $s \longleftrightarrow j\omega$

Frequency response function

Transfer function: $H(s)$

Definition: $H_F(\omega) = H(j\omega)$ (frequency response function)

Then H_F is the Fourier-transform of the impulse response function (h) since:

$$H_F(\omega) = \int_0^{\infty} h(t)e^{-i\omega t} dt = \int_{-\infty}^{\infty} h(t)e^{-i\omega t} dt$$

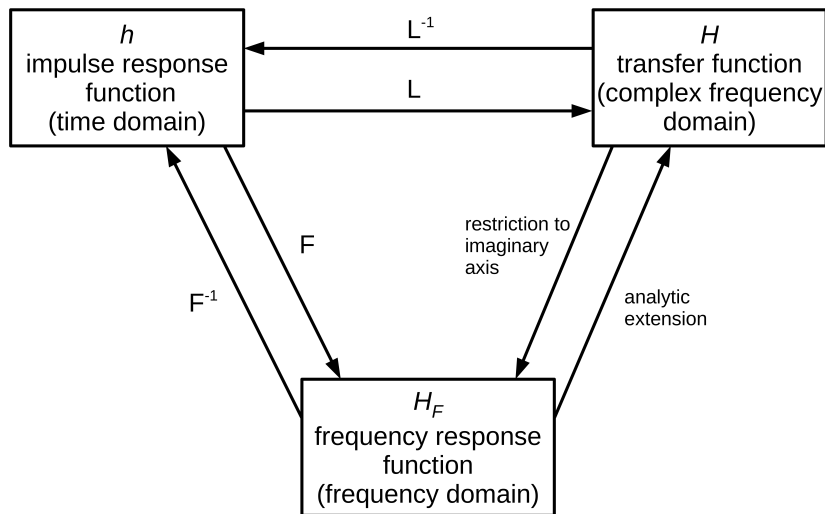
H_F is the restriction of H to the imaginary axis

Question: Can we compute H from the restriction on the complex plane, where the Laplace-transform is defined?

Answer: Using the fact that the transfer function is *analytic*, the computation is the following, if the poles of H are on the left half-plane:

$$H(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{H_F(\omega)}{s - i\omega} d\omega$$

Time, operator, and frequency-domains



Response of stable LTI systems to periodic inputs

Theorem: Let $H(s)$ be the transfer function of an asymptotically stable LTI system, and $\omega > 0$. Then the response of the system to the input $u(t) = u_0 \sin(\omega t)$ is of the following form:

$$y(t) = u_0 \operatorname{Re}(H_F(\omega)) \sin(\omega t) + u_0 \operatorname{Im}(H_F(\omega)) \cos(\omega t)$$

(we do not prove)

Remarks:

- It is visible that the output is also periodic with a period $T = \frac{2\pi}{\omega}$ equal to the period of the input.
- The theorem is still valid if the transfer function has purely imaginary poles of the form $i\hat{\omega}$, but $\omega/\hat{\omega} \notin \mathbb{Z}$.

Response of stable LTI systems to periodic inputs

transfer function: $G(j\omega)$, $(G(s))$

$$u(t) = u_0 \sin(\omega t + \alpha)$$

$$y(t) \longrightarrow y_0 \sin(\omega t + \beta)$$

gain: $k = \left| \frac{y_0}{u_0} \right| = |G(j\omega)|$ (frequency dependent!)

phase: $\phi = \beta - \alpha = \angle G(j\omega[\text{rad}])$ (frequency dependent!)

E.g. let $G(j\omega) = a + bj$

$$|G(j\omega)| = \sqrt{(a^2 + b^2)}, \quad \angle G(j\omega) = \arctan(b/a)$$

Gain in time and frequency domains

$$u(t) = a_0 \sin(\omega t), \quad y(t) = a_1 \sin(\omega t + \phi)$$

$$U(s) = \frac{a_0 \omega}{s^2 + \omega^2}, \quad Y(s) = \frac{a_1 (s \sin(\phi) + \omega \cos(\phi))}{s^2 + \omega^2}$$

$$|G(j\omega)| = \frac{|Y(j\omega)|}{|U(j\omega)|} = \left| \frac{a_1 (j\omega \sin(\phi) + \omega \cos(\phi))}{a_0 \omega} \right| = \left| \frac{a_1}{a_0} \right|$$

$$\angle G(j\omega) = \arctan \left(\frac{\omega \sin(\phi)}{\omega \cos(\phi)} \right) = \phi$$

Example: RLC circuit – 5

Transfer function: $C(sI - A)^{-1}B = \frac{100}{s^2 + 10s + 100} = \frac{100}{(j\omega)^2 + 10(j\omega) + 100}$

- $f = 0$ Hz, $\omega = 0$ rad/s, $G(j\omega) = 1 + 0j$, $|G(j\omega)| = 1$, $\phi = 0$ rad
- $f = 1$ Hz, $\omega = 6.2832$ rad/s, $G(j\omega) = 0.7952 - 0.8256j$,
 $|G(j\omega)| = 1.1463$, $\phi = -0.8041$ rad
- $f = 5$ Hz, $\omega = 31.4159$ rad/s, $G(j\omega) = -0.1002 - 0.0355j$,
 $|G(j\omega)| = 0.1063$, $\phi = 0.3404$ rad
- $f = 10$ Hz, $\omega = 62.8319$ rad/s, $G(j\omega) = -0.0253 - 0.004j$,
 $|G(j\omega)| = 0.0256$, $\phi = 0.1619$ rad

Gain of transfer functions

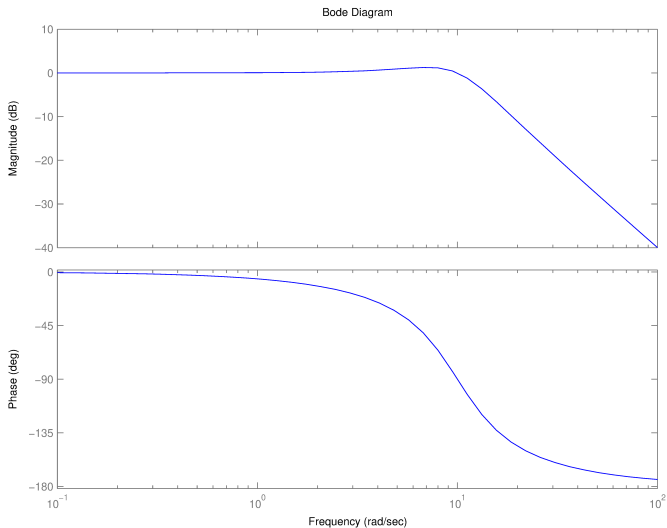
$$A = \left| \frac{y_0}{u_0} \right|$$

in dB:

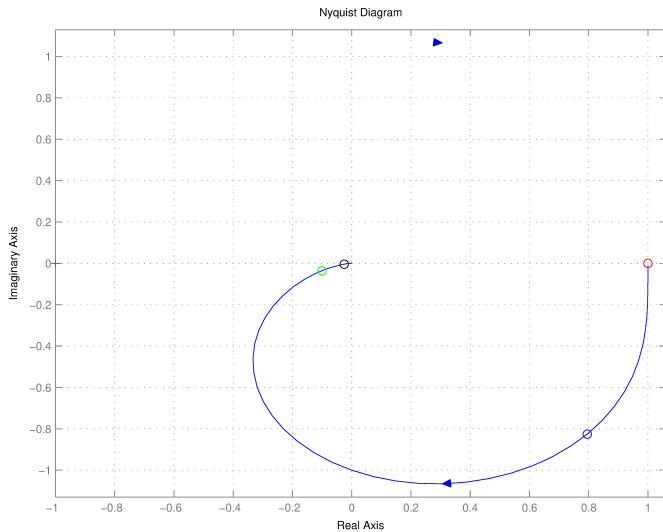
$$A_d = 20 \cdot \log_{10}(A) \text{ [dB]}$$

- $|G(j\omega)| = 1, A_d = 0 \text{ dB}$
- $|G(j\omega)| = 1.1463, A_d = 1.1860 \text{ dB}$
- $|G(j\omega)| = 0.1063, A_d = -19.4693 \text{ dB}$
- $|G(j\omega)| = 0.0256, A_d = -31.8352 \text{ dB}$

Bode-diagram



Nyquist-diagram



Bandwidth of SISO systems

Bandwidth: Frequency, where $|G(j\omega)|$ first crosses the value $1/\sqrt{2}$ (≈ -3 dB) from above

Example: RLC circuit

$$y = u_C, \omega_c \approx 2.03 \text{ Hz}$$

Transfer function of MIMO systems

$$u \in \mathbb{R}^m, y \in \mathbb{R}^r$$

$$Y(s) = H(s)U(s),$$

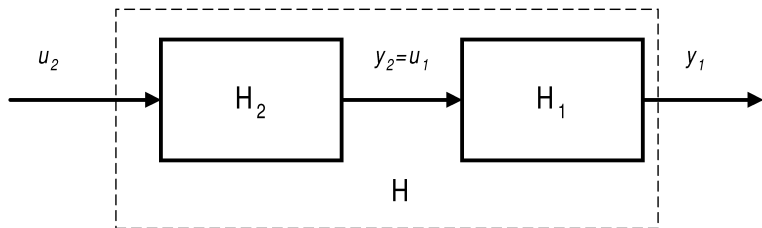
$$H(s) = \begin{bmatrix} h_{11}(s) & \dots & h_{1m}(s) \\ h_{r1}(s) & \dots & h_{rm}(s) \end{bmatrix} \in \mathbb{C}^{r \times m}$$

$$\text{Pl. RLC-circuit, } u = u_{in}, y = [i \quad u_C]^T$$

$$H(s) = \begin{bmatrix} \frac{10s}{s^2+10s+100} \\ \frac{100}{s^2+10s+100} \end{bmatrix}$$

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Serial interconnection of subsystems

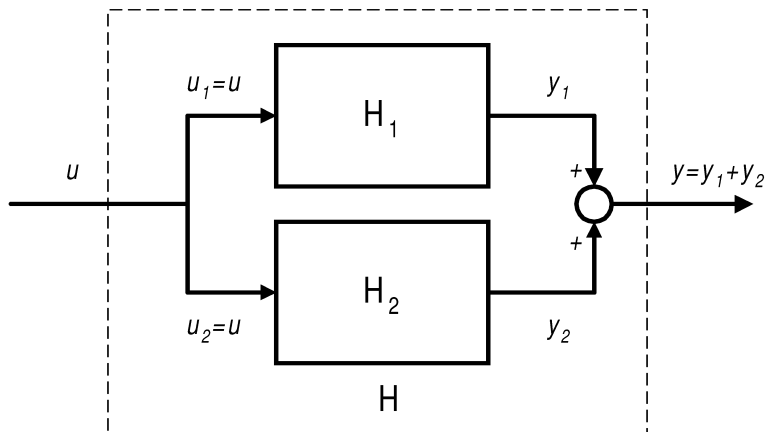


$$H(s) = H_1(s) \cdot H_2(s)$$

i.e.

$$h(t) = (h_1 * h_2)(t)$$

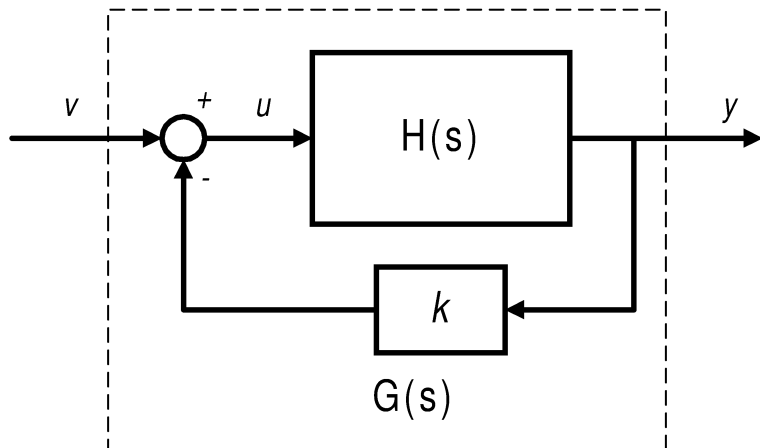
Parallel interconnection of subsystems



$$H(s) = H_1(s) + H_2(s)$$

$$h(t) = h_1(t) + h_2(t)$$

Proportional negative feedback



$$G(s) = \frac{H(s)}{1 + k \cdot H(s)}$$

Negative feedback – example

Original system:

$$H(s) = \frac{1}{s - 1}, \quad (\text{unstable})$$

Feedback system:

$$G(s) = \frac{1}{s + k - 1}$$

stable, if $k > 1$

High gain output feedback

$$H(s) = \frac{b(s)}{a(s)}$$

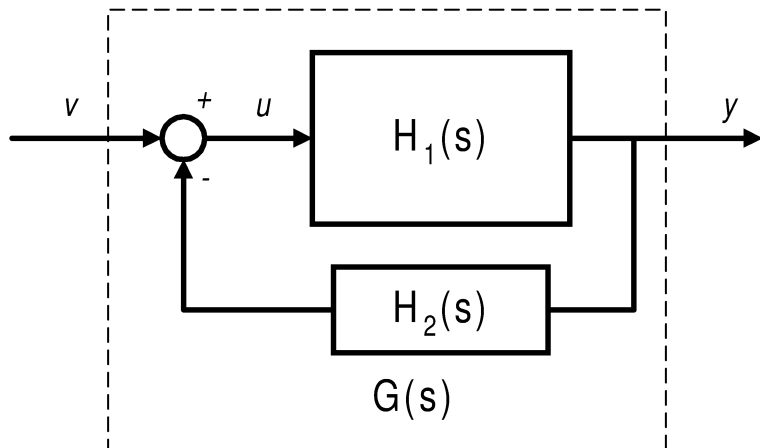
Transfer function of the feedback system:

$$G(s) = \frac{b(s)}{a(s) + k \cdot b(s)} = \frac{n(s)}{d(s)}$$

For $k \rightarrow \infty$, $d(s) \rightarrow b(s)$, i.e. by increasing the feedback gain, the poles of the feedback system converge to the zeros of the original system.

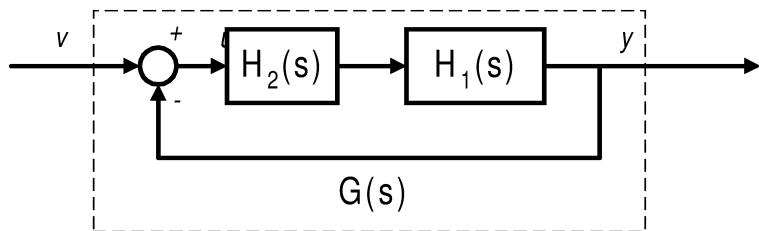
Minimum phase systems: Such systems where the real part of each zero is negative. (They can be stabilized by high gain feedback.)

General negative feedback – 1



$$G(s) = \frac{H_1(s)}{1 + H_1(s)H_2(s)}$$

General negative feedback – 2



$$G(s) = \frac{H_1(s)H_2(s)}{1 + H_1(s)H_2(s)}$$

Summary

- SISO transfer functions (TFs) are complex numbers (with absolute value and angle) at any given s
- frequency domain interpretation: assuming periodic (sinusoidal) input, $s = j\omega$
- absolute value of TF: gain (ratio of O/I amplitudes) at a given frequency
- angle of TF: phase shift at a given frequency
- visualization: Bode diagram, Nyquist diagram
- overall transfer functions were computed for different basic interconnection of subsystems