Computer Controlled Systems
Lecture 4

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Contents

Introductory example

Consider the following SISO CT-LTI system withe realization (A,B,C)

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$$

The model is *observable* but it is *not controllable*.
Question: Can the model be written in a new coordinates system, such that the new model is both observable and controllable? (and what are the conditions / consequences?)
Transfer function:

$$H(s) = \frac{2s^2 + 4s}{s^3 + 2s^2 - s}$$

Introduction – 1

- For a given (SISO) transfer function $H(s) = \frac{b(s)}{a(s)}$, the state space model (A, B, C, D) is called *an n-th order realization* (or *n-dimensional realization*) if $H(s) = C(sI - A)^{-1}B + D$, where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$, $C \in \mathbb{R}^{1\times n}$, $D \in \mathbb{R}$.
  (The state space repr. for a given transfer function is **not unique**).
- An *n-th order state space realization* (A, B, C, D) of a given transfer function $H(s)$ is called *minimal*, if there exist no other realization with a smaller state space dimension (i.e., with a smaller A matrix)
- An n-th order state *space model* (A, B, C, D) is called *jointly controllable and observable* if both $\mathcal{O}_n$ and $\mathcal{C}_n$ are full-rank matrices.

Introduction – 2

- The transfer function is invariant for state transformations
- The roots of the transfer function's denominator are the eigenvalues of matrix A ($a(s)$ is the characteristic polynomial of A)
- For a given transfer function $H(s)$, any two arbitrary jointly controllable and observable realizations (A_1, B_1, C_1) and (A_2, B_2, C_2) are connected to each other by the following coordinates transformation

$$T = \mathcal{O}^{-1}(C_1, A_1)\mathcal{O}(C_2, A_2) = \mathcal{C}(A_1, B_1)\mathcal{C}^{-1}(A_2, B_2)$$

(without proof)

Introduction – 3

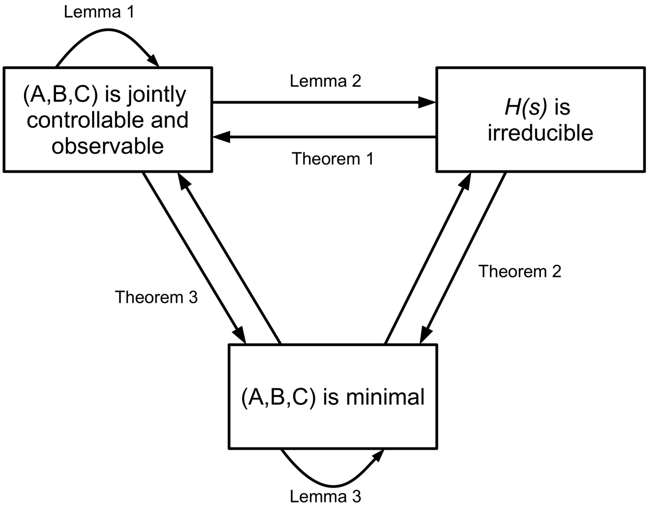
Matrix polynomials:

$$p(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0, \quad x \in \mathbb{R}$$
$$p(A) = c_n A^n + c_{n-1} A^{n-1} + \cdots + c_1 A + c_0 I$$

important properties:

- a matrix polynomial commutes with any power of the argument matrix, namely: $A^i P(A) = P(A) A^i$
- eigenvalues: $\lambda_i[P(A)] = P(\lambda_i[A])$
- Cayley-Hamilton theorem: every $n \times n$ matrix is a root of its own characteristic polynomial ($p(x) = \det(A - xI)$)

Overview – 1



equivalent state space and I/O model properties

Overview – 2

Consider **SISO CT-LTI systems** with realization (A, B, C)

- Joint controllability and observability is a **system property**
- Equivalent necessary and sufficient conditions
- Minimality of SSRs
- Irreducibility of the transfer function

Assumptions

- We consider SISO systems (scalar input/output)
- We assume that the transfer function is strictly proper, i.e.

$$H(s) = \frac{b(s)}{a(s)},$$

where $a(s) = s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n$, and
$b(s) = b_1 s^{n-1} + \ldots + b_{n-1}s + b_n$
Remark: proper transfer functions (where the degree of $a(s)$ and $b(s)$ are equal) can be written in the form $H(s) = \frac{b(s)}{a(s)} + D$, where $\frac{b(s)}{a(s)}$ is strictly proper, and we can define a transformed output $\hat{y} = y - Du$ for which

$$\hat{Y}(s) = \frac{b(s)}{a(s)} U(s)$$

Hankel matrices

- A *Hankel matrix* is a block matrix of the following form

$$H[1, n-1] = \begin{bmatrix} CB & CAB & \ldots & CA^{n-1}B \\ CAB & CA^2B & \ldots & CA^nB \\ \vdots & \vdots & \vdots\;\vdots\;\vdots & \vdots \\ CA^{n-1}B & CA^nB & \ldots & CA^{2n-2}B \end{bmatrix}$$

- It contains *Markov parameters* CA^iB that are invariant under state transformations.

Lemma 1

Lemma (1)

If we have a system with transfer function $H(s) = \frac{b(s)}{a(s)}$ and there is an n-th order realization (A, B, C) which is jointly controllable and observable, then all other n-th order realizations are jointly controllable and observable.

Proof

$$\mathcal{O}(C, A) = \begin{bmatrix} C \\ CA \\ . \\ . \\ . \\ CA^{n-1} \end{bmatrix}, \quad \mathcal{C}(A, B) = \begin{bmatrix} B & AB & A^2B & . & . & . & A^{n-1}B \end{bmatrix}$$

$$H[1, n-1] = \mathcal{O}(C, A)\mathcal{C}(A, B)$$

Controller form realization

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

with

$$A_c = \begin{bmatrix} -a_1 & -a_2 & . & . & . & -a_n \\ 1 & 0 & . & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & 1 & 0 \end{bmatrix}, \quad B_c = \begin{bmatrix} 1 \\ 0 \\ . \\ . \\ . \\ 0 \end{bmatrix}$$

$$C_c = \begin{bmatrix} b_1 & b_2 & . & . & . & b_n \end{bmatrix}$$

with the coefficients of the polynomials
$a(s) = s^n + a_1 s^{n-1} + ... + a_{n-1}s + a_n$ and $b(s) = b_1 s^{n-1} + ... + b_{n-1}s + b_n$
that appear in the transfer function $H(s) = \frac{b(s)}{a(s)}$

Observer form realization

$$\dot{x}(t) = A_o x(t) + B_o u(t)$$
$$y(t) = C_o x(t)$$

where

$$A_o = \begin{bmatrix} -a_1 & 1 & 0 & \dots & 0 \\ -a_2 & 0 & 1 & \dots & 0 \\ \vdots & & & & \\ -a_{n-1} & 0 & 0 & \dots & 1 \\ -a_n & 0 & 0 & \dots & 0 \end{bmatrix}, \quad B_o = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix}$$

$$C_o = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \end{bmatrix},$$

with the coefficients of the polynomials
$a(s) = s^n + a_1 s^{n-1} + ... + a_{n-1}s + a_n$ and $b(s) = b_1 s^{n-1} + ... + b_{n-1}s + b_n$
that appear in the transfer function $H(s) = \frac{b(s)}{a(s)}$

Definitions

Definition (Relative prime polynomials)

Two polynomials $a(s)$ and $b(s)$ are *coprimes* (or relative primes) if $a(s) = \prod(s - \alpha_i)$; $b(s) = \prod(s - \beta_j)$ and $\alpha_i \neq \beta_j$ for all i, j.
In other words: the polynomials have no common roots.

Definition (Irreducible transfer function)

A transfer function $H(s) = \frac{b(s)}{a(s)}$ is called to be *irreducible* if the polynomials $a(s)$ and $b(s)$ are relative primes.

Lemma 2

Lemma (2)

An n-dimensional controller form realization with transfer function $H(s) = \frac{b(s)}{a(s)}$ (where $a(s)$ is an n-th order polynomial) is jointly controllable and observable if and only if $a(s)$ and $b(s)$ are relative primes (i.e., $H(s)$ is irreducible).

Proof

- A controller form realization is controllable and

$$\mathcal{O}_c = \tilde{I}_n b(A_c)$$

$$\tilde{I}_n = \begin{bmatrix} 0 & . & . & 1 \\ 0 & . & 1 & 0 \\ . & . & . & . \\ 1 & 0 & . & 0 \end{bmatrix} \in \mathbb{R}^{n \times n}$$

- Non-singularity of $b(A_c)$

Proof of Lemma 2. – 1

$$\tilde{I}_n = \begin{bmatrix} e_n & e_{n-1} & . & . & e_1 \end{bmatrix} = \begin{bmatrix} e_n^T \\ e_{n-1}^T \\ . \\ . \\ . \\ e_1^T \end{bmatrix} \quad , \quad e_i = \begin{bmatrix} 0 \\ . \\ . \\ 0 \\ 1 \\ 0 \\ . \\ . \end{bmatrix} \leftarrow i.$$

$$A_c = \begin{bmatrix} -a_1 & -a_2 & . & . & . & -a_n \\ 1 & 0 & . & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & 1 & 0 \end{bmatrix} \quad , \quad e_i^T A_c = \begin{cases} [-a_1 \quad -a_2 \quad \ldots \quad -a_n] \\ e_{i-1}^T \end{cases}$$

Proof of Lemma 2. – 2

- Computation of the observability matrix $\mathcal{O}_c = \tilde{I}_n b(A_c) \in \mathbb{R}^{n \times n}$
- 1st row:

$$e_n^T b(A_c) = e_n^T b_1 A_c^{n-1} + \ldots + e_n^T b_{n-1} A_c + e_n^T b_n I_n$$

n-th term: $[0 \ \ldots \ 0 \ b_n]$
$(n-1)$-th term: $b_{n-1} e_n^T A_c = b_{n-1} e_{n-1}^T = [0 \ \ldots \ b_{n-1} \ 0]$
...

$$e_n^T b(A_c) = [b_1 \ \ldots \ b_{n-1} \ b_n] = C_c$$

- 2nd row:

$$e_{n-1}^T b(A_c) = e_n^T A_c b(A_c) = e_n^T b(A_c) A_c \quad \Rightarrow \quad e_{n-1}^T b(A_c) = C_c A_c$$

- and so on ...

Proof of Lemma 2. – 3

$\mathcal{O}_c$ is nonsingular

- iff $b(A_c)$ is nonsingular because matrix $\tilde{I}_n$ is always nonsingular
- $b(A_c)$ is nonsingular iff $det(b(A_c)) \neq 0$
  which depends on the eigenvalues of $b(A_c)$ matrix
- the eigenvalues of the matrix $b(A_c)$ are $b(\lambda_i), \quad i = 1, 2, ..., n$
  λ_i is an eigenvalue of A_c, i.e a root of $a(s) = det(sI - A)$

$$det(b(A_c)) = \prod_{i=1}^{n} b(\lambda_i) \neq 0$$

$$\Updownarrow$$

$a(s)$ and $b(s)$ have no common roots, i.e. they are relative primes

Minimal realization conditions – 1

Theorem (1)

$H(s) = \frac{b(s)}{a(s)}$ (where $a(s)$ is an n-th order polynomial) is irreducible if and only if all of its n-th order realizations are jointly controllable and observable.

Proof: combine Lemma 1. and 2.

- We assume that any nth order realization $H(s)$ is jointly controllable and observable $\Longrightarrow$ A controller form is jointly controllable and observable $\Longrightarrow$ $H(s)$ is irreducible (Lemma 2)
- We assume that $H(s)$ is irreducible $\Longrightarrow$ the controller form realization is jointly controllable and observable (Lemma 2) $\Longrightarrow$ Any nth order realization is jointly controllable and observable (Lemma 1)

Minimal realization conditions – 2

Definition (Minimal realization)

An n-dimensional realization (A, B, C) of the transfer function $H(s)$ is minimal if one cannot find another realization of $H(s)$ with dimension less than n.

Theorem (2)

$H(s) = \frac{b(s)}{a(s)}$ is irreducible iff any of its realization (A, B, C) is minimal where $H(s) = C(sI - A)^{-1}B$

Proof: by contradiction

- We assume that $H(s)$ is irreducible, but there exists an nth order realization, which is not minimal $\Longrightarrow$ there exists an mth $(m < n)$ order realization $(\bar{A}, \bar{B}, \bar{C})$ of $H(s)$ $\Longrightarrow$ from this realization we can obtain the transfer function $\bar{H}(s)$, for which the order of its denominator m, which is a contradiction (since $H(s)$ is reducible).
- We assume that the nth order realization (A, B, C) is minimal, but $H(s) = C(sI - A)^{-1}B$ is reducible $\Longrightarrow$ From the simplified transfer function one can obtain an mth order realization, such that $m < n$, that is a contradiction.

Minimal realization conditions – 3

Theorem (3)

A realization (A, B, C) is minimal iff the system is jointly controllable and observable.

Proof: Combine Theorem 1 and Theorem 2 .

Lemma (3)

Any two minimal realizations can be connected by a unique similarity transformation (which is invertible).

Proof: (Just the idea of it)

$$T = \mathcal{O}^{-1}(C_1, A_1)\mathcal{O}(C_2, A_2) = \mathcal{C}(A_1, B_1)\mathcal{C}^{-1}(A_2, B_2)$$

exists and it is invertible: this is used as a transformation matrix.

Decomposition of uncontrollable systems

We assume that (A, B, C) is not controllable. Then, there exists an invertible transformation T such that the transformed system in the new coordinates system ($\bar{x} = Tx$) will have the form

$$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{bmatrix} = \begin{bmatrix} A_c & A_{12} \\ 0 & A_{\bar{c}} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \begin{bmatrix} B_c \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} C_c & C_{\bar{c}} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}$$

and

$$H(s) = C_c(sI - A_c)^{-1}B_c$$

$\bar{x}_2$ is not affected by u, and does not depend on $\bar{x}_1$.

Controllability decomposition – example

Matrices of the state-space :

$$A = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad D = 0$$

Controllability matrix:

$$\mathcal{C}_2 = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

Transformation:

$$T^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad T = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

The transformed model:

$$\bar{A} = \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix}, \ \bar{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} 2 & 1 \end{bmatrix}$$

Decomposition of unobservable systems

We assume that (A, B, C) is not observable. Then there exists an invertible matrix transformation T, such that the transformed system in the new coordinates system ($\bar{x} = Tx$) will have the form

$$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{bmatrix} = \begin{bmatrix} A_o & 0 \\ A_{21} & A_{\bar{o}} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \begin{bmatrix} B_o \\ B_{\bar{o}} \end{bmatrix} u$$

$$y = \begin{bmatrix} C_o & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}$$

and

$$H(s) = C_o(sI - A_o)^{-1} B_o$$

$\bar{x}_2$ *itself is not observed and it does not affect* $\bar{x}_1$ *(which is observed).*

Observability decomposition – example

Matrices of the state-space model:

$$A = \begin{bmatrix} 1 & 2 \\ -2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad D = 0$$

Observability matrix:

$$\mathcal{O}_2 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

Transformation:

$$T = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, \quad T^{-1} = \begin{bmatrix} 1 & -0.5 \\ 0 & 0.5 \end{bmatrix}$$

The transformed model:

$$\bar{A} = \begin{bmatrix} -1 & 0 \\ -4 & 1 \end{bmatrix}, \; \bar{B} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

General decomposition theorem

Given an (A, B, C) SSR, it is always possible to transform it to another realization $(\overline{A}, \overline{B}, \overline{C})$ with partitioned state vector and matrices

$$\overline{x} = \begin{bmatrix} \overline{x}_{co} & \overline{x}_{c\overline{o}} & \overline{x}_{\overline{c}o} & \overline{x}_{\overline{co}} \end{bmatrix}^T$$

$$\overline{A} = \begin{bmatrix} \overline{A}_{co} & 0 & \overline{A}_{13} & 0 \\ \overline{A}_{21} & \overline{A}_{c\overline{o}} & \overline{A}_{23} & \overline{A}_{24} \\ 0 & 0 & \overline{A}_{\overline{c}o} & 0 \\ 0 & 0 & \overline{A}_{43} & \overline{A}_{\overline{co}} \end{bmatrix} \quad \overline{B} = \begin{bmatrix} \overline{B}_{co} \\ \overline{B}_{c\overline{o}} \\ 0 \\ 0 \end{bmatrix}$$

$$\overline{C} = \begin{bmatrix} \overline{C}_{co} & 0 & \overline{C}_{\overline{c}o} & 0 \end{bmatrix}$$

General decomposition theorem

The partitioning defines subsystems

- *Controllable and observable subsystem*: $(\overline{A}_{co}, \overline{B}_{co}, \overline{C}_{co})$ is minimal, i.e. $\overline{n} \leq n$ and

$$H(s) = \overline{C}_{co}(s\overline{I} - \overline{A}_{co})^{-1}\overline{B}_{co} = C(sI - A)^{-1}B$$

- *Controllable subsystem*

$$\left(\begin{bmatrix} \overline{A}_{co} & 0 \\ \overline{A}_{21} & \overline{A}_{c\overline{o}} \end{bmatrix} , \begin{bmatrix} \overline{B}_{co} \\ \overline{B}_{c\overline{o}} \end{bmatrix} , \begin{bmatrix} \overline{C}_{co} & 0 \end{bmatrix} \right)$$

- *Observable subsystem*

$$\left(\begin{bmatrix} \overline{A}_{co} & \overline{A}_{13} \\ 0 & \overline{A}_{\overline{c}o} \end{bmatrix} , \begin{bmatrix} \overline{B}_{co} \\ 0 \end{bmatrix} , \begin{bmatrix} \overline{C}_{co} & \overline{C}_{\overline{c}o} \end{bmatrix} \right)$$

- *Uncontrollable and unobservable subsystem*

$$([\overline{A}_{\overline{co}}] \ , \ [0] \ , \ [0])$$

Introductory example – review

Consider the following SISO CT-LTI system withe realization (A,B,C)

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$$

The model is *observable* but it is *not controllable*.
Its transfer function and its simplified form:

$$H(s) = \frac{2s^2 + 4s}{s^3 + 2s^2 - s} = \frac{2s + 4}{s^2 + 2s - 1}$$

Its minimal state space realization (eq. controller form):

$$\bar{A} = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} 2 & 4 \end{bmatrix}$$

Summary

- joint controllability and observability of (A, B, C) has important consequences, since it is equivalent to:
  - a state space realization with the minimum number of state variables (minimal realization, i.e., A cannot be smaller)
  - $H(s) = C(sI - A)^{-1}B = \frac{b(s)}{a(s)}$ is irreducible
- non-controllable and/or non-observable state space models can be transformed such that the non-controllable / non-observable states are clearly visible in the new coordinates
- it's easy to determine a minimal realization from a non-controllable/non-observable SS model (simplification of the transfer function, canonical realization)