Computer Controlled Systems (Introduction to systems and control theory) Lecture 3

Gábor Szederkényi

Pázmány Péter Catholic University
Faculty of Information Technology and Bionics

e-mail: szederkenyi@itk.ppke.hu

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Contents

Needed from mathematics

- matrices: row/column rank, image, left/right kernel, determinant, characteristic polynomial
- matrix polynomials, Cayley-Hamilton theorem
- quadratic forms

Revision from previous lecture

- **system**: operator (input signals $\longrightarrow$ output signals)
- **LTI models**: I/O: $H(s)$, state space: (A, B, C, D)
- **states**: form the state space; knowing the model, input, and initial state the future states and outputs can be computed

Brief problem statements of observability and controllability

General form of SS models – revision

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) \quad , \quad x(t_0) = x(0)\\ y(t) &= Cx(t)\end{aligned}$$

- signals: $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^p$, $u(t) \in \mathbb{R}^r$
- system parameters: $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times r}$, $C \in \mathbb{R}^{p\times n}$ (assumption without loss of generality: $D = 0$)

Studied system properties:

- **observability**: determining the initial condition (we need state information from the measurements (output) knowing the model)
- **controllability**: setting the initial condition (we want to influence (change) the state with appropriate input knowing the model)

Observability of LTI systems – 1

Problem formulation

Given:

- a state space model (A, B, C) $(D = 0)$
- input u
- measured values y on a finite time horizon

To be computed:
The value of state vector x on a finite time horizon

It is sufficient to compute: $x(t_0) = x_0$

Definition. *The system (A, B, C) (or, equivalently, the pair (A, C)) is observable, if $x(t_0)$ can be determined from a finite measurement of y.*

Observability – example 1

We consider the known RLC circuit. We measure the voltage of the capacitor (u_C). We want to obtain the value of the current (i).

$$x = \begin{bmatrix} i \\ u_C \end{bmatrix}, \quad u = u_{be}, \quad y = x_2$$

$$A = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

Observability – example 2

Elementary acceleration model (without friction, air resistance etc.):

$$F = m \cdot a = m \cdot \ddot{x}_1$$

in state space form:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u, \end{aligned}$$

where x_1 is the position, x_2 is the velocity, and $u = \frac{F}{m}$.

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Problems/tasks:

a) Can we determine the velocity if the position is measured only? (i.e. $C = [1 \; 0]$)

b) Can we determine the position if the velocity is measured only? (i.e. $C = [0 \; 1]$)

Observability of LTI systems – 2

Necessary and sufficient condition.

A state space model (A, B, C) is *observable* **if and only if** the observability matrix $\mathcal{O}_n$ is full-rank.

$$\mathcal{O}_n = \begin{bmatrix} C \\ CA \\ \cdot \\ \cdot \\ \cdot \\ CA^{n-1} \end{bmatrix}$$

Observability of LTI systems – 3

Proof: (by construction)

$$
\begin{aligned}
y &= Cx\\
\dot{y} &= C\dot{x} = CAx + CBu\\
\ddot{y} &= C\ddot{x} = CA(Ax + Bu) + CB\dot{u} = CA^2x + CABu + CB\dot{u}\\
&.\\
&.\\
y^{(n-1)} &= Cx^{(n-1)} = CA^{n-1}x + CA^{n-2}Bu + ... + CABu^{(n-3)} + CBu^{(n-2)}
\end{aligned}
$$

$$
\begin{bmatrix} y \\ \dot{y} \\ \ddot{y} \\ . \\ . \\ . \\ y^{(n-1)} \end{bmatrix}
=
\begin{bmatrix} C \\ CA \\ CA^2 \\ . \\ . \\ . \\ CA^{n-1} \end{bmatrix} x +
\begin{bmatrix}
0 & 0 & . & . & . & 0 \\
CB & 0 & . & . & . & 0 \\
CAB & CB & 0 & . & . & 0 \\
. & . & . & . & . & . \\
. & . & . & . & . & . \\
. & . & . & . & . & . \\
CA^{n-2}B & CA^{n-3}B & . & . & CB & 0
\end{bmatrix}
\begin{bmatrix} u \\ \dot{u} \\ \ddot{u} \\ . \\ . \\ . \\ u^{(n-1)} \end{bmatrix}
$$

Observability of LTI systems – 4

$$\mathcal{Y}(t) = \mathcal{O}_n x(t) + \mathcal{T}\mathcal{U}(t)$$

Expressing $x(t)$

$$x(t) = \mathcal{O}_n^{-1}(\mathcal{Y}(t) - \mathcal{T}\mathcal{U}(t)),$$

where $\mathcal{O}^{-1}$ denotes the (generalized) inverse of $\mathcal{O}_n$

$x(t)$ **can be uniquely determined if and only if** $\mathit{rank}\ \mathcal{O}_n(A, C) = n$.

Controllability of LTI systems – 1

Problem formulation
Given:

- a state space mode (A, B, C)
- **initial condition** $x(0)$, and $x(T) \neq x(0)$ desired **final state**

To be computed:
an appropriate u input signal, which drives the system from state $x(t_1)$ to $x(t_2)$ **in finite time**.

Definition. *The system (A, B, C) (or, equivalently, the pair (A, B)) is controllable if, given a finite duration $T > 0$ and two arbitrary points $x_0, x_T \in \mathbb{R}^n$, there exists an appropriate input u such that for initial condition $x(0) = x_0$, the value of the state vector at time T is $x(T) = x_T$.*

Controllability – example

System: RLC circuit $i(0) = 1A$, $u(0) = 0V$
Does there exist an input voltage function u_{be}, such that we have $i(t_1) = 5A$, $u(t_1) = 10V$, and $t_1 < M < \infty$?

Controllability – example 2

Elementary acceleration model again

$$\dot{x} = Ax + Bu$$

where x_1 is the position, x_2 is the velocity, and $u = \frac{F}{m}$
matrices of the SS model:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Problem/task:
Compute an acceleration command such that the speed is exactly $x_2 = 30m/s$ at distance $x_1 = 200m$?

Controllability of LTI systems – 2

Necessary and sufficient condition
A state space model with matrices (A, B, C)

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

is controllable **if and only if**, the controllability matrix $\mathcal{C}_n$ is of full-rank

$$\mathcal{C}_n = \left[\begin{array}{cccccc} B & AB & A^2B & . & . & A^{n-1}B \end{array} \right]$$

Controllability of LTI systems – 3

Proof: (by construction)
$\int_{-\infty}^{\infty} f(t)\delta'(t)dt = -f'(0)$
$\int_{-\infty}^{\infty} f(t)\delta^{(n)}(t)dt = (-1)^n f^{(n)}(0)$
$f(\tau) = e^{-A\tau}, \quad f'(\tau) = -Ae^{-A\tau}$
$f^{(n)}(\tau) = (-1)^n A^n e^{-A\tau}$
Input: linear combination of Dirac-δ and its time derivatives.
$u(t) = g_1\delta(t) + g_2\dot{\delta}(t) + ... + g_n\delta^{(n-1)}(t)$
According to the principle of superposition:

$$x(0_+) = x(0_-) + g_1 h(0_-) + g_2\dot{h}(0_-) + ... + g_n h^{(n-1)}(0_-)$$
$$x(0_+) = x(0_-) + g_1 B + g_2 AB + ... + g_n A^{n-1} B$$

Controllability of LTI systems – 4

Assuming that $x(0_-) = 0$ we get:

$$x(0_+) = \left[\begin{array}{ccccccc} B & AB & A^2B & . & . & . & A^{n-1}B \end{array} \right] \left[\begin{array}{c} g_1 \\ g_2 \\ . \\ . \\ . \\ g_n \end{array} \right]$$

for an arbitrary final state value $x(0_+)$ there exists a unique weighting vector $[g_1 ... g_n]^T$ if and only if *rank* $\mathcal{C}_n(A, B) = n$.

Diagonal realization

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

where

$$\dot{x} = \begin{bmatrix} \lambda_1 & . & . & . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & . & . & . & \lambda_n \end{bmatrix} x + \begin{bmatrix} b_1 \\ . \\ . \\ . \\ b_n \end{bmatrix} u$$
$$y = \begin{bmatrix} c_1 & . & . & . & c_n \end{bmatrix} x$$

Controllability in case of a diagonal realization

Controllability matrix

$$\mathcal{C}_n = \begin{bmatrix} B & AB & . & . & A^{n-1}B \end{bmatrix} = \begin{bmatrix} b_1 & \lambda_1 b_1 & \lambda_1^2 b_1 & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ b_n & \lambda_n b_n & \lambda_n^2 b_n & . & . \end{bmatrix} =$$

$$= \begin{bmatrix} b_1 & . & . & . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & . & . & . & b_n \end{bmatrix} \begin{bmatrix} 1 & \lambda_1 & . & . & \lambda_1^{n-1} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ 1 & \lambda_n & . & . & \lambda_n^{n-1} \end{bmatrix}$$

This matrix is the so-called *Vandermonde-matrix*, which is nonsingular if $\lambda_i \neq \lambda_j \quad (i \neq j)$.

$$rank\ \mathcal{C}_n = n \quad \Leftrightarrow \quad det\ \mathcal{C}_n \neq 0$$
$$det\ \mathcal{C}_n = \prod_i b_i \prod_{i<j} (\lambda_i - \lambda_j)$$

Transfer function of a diagonal SISO realization

$$H(s) = C(sI - A)^{-1}B = \sum_{i=1}^{n} \frac{c_i b_i}{s - \lambda_i} = \frac{b(s)}{a(s)}$$

where I is the unit matrix.

If $c_j = 0$ or $b_k = 0$ for a given j or k, then the transfer function can be rewritten by using a smaller number of partial fractions.

$$H(s) = \sum_{i=1}^{\overline{n}} \frac{c_i b_i}{s - \lambda_i} = \frac{b(s)}{a(s)} \quad , \quad \overline{n} < n$$

Computing a realizable (smooth) input for a target state

Given: A, B, $x(0)$ (initial state), $x(\bar{t})$ (target state)
To be determined: u, $\bar{t}$ (finite)
Assumption: input is in the form $u(t) = B^T e^{A^T(\bar{t}-t)} z$, where $z \in \mathbb{R}^n$ ($z =$?)

$$x(\bar{t}) = e^{A\bar{t}} x(0) + \int_0^{\bar{t}} e^{A(\bar{t}-\tau)} B B^T e^{A^T(\bar{t}-\tau)} \cdot z d\tau$$

Let $\xi = \bar{t} - \tau$, then:

$$x(\bar{t}) = e^{A\bar{t}} x(0) + \underbrace{\left[\int_0^{\bar{t}} e^{A\xi} B B^T e^{A^T \xi} d\xi \right]}_{G_C(\bar{t})} \cdot z$$

From this, the input parameters can be expressed as

$$z = G_c^{-1}(\bar{t}) \left(x(\bar{t}) - e^{A\bar{t}} x(0) \right), \quad \text{provided that } G_c^{-1}(\bar{t}) \text{ exists for some } \bar{t}$$

The controllability Gramian

$$G_C(t) = \int_0^t e^{A\tau} B B^T e^{A^T \tau} d\tau$$

is the controllability Gramian

The following is true:

The controllability matrix is of full rank if and only if $G_C(t)$ is positive definite (and therefore, invertible) for some $t \geq 0$.

controllability $\Longrightarrow$ a smooth input can be computed to arbitrarily change the state in finite time

Remark on the powers of A

- **Q**: Why is it that there is no need for higher powers of A than A^{n-1} in the controllability/observability matrices?
- **A**: It follows from the Cayley-Hamilton theorem that A^n, A^{n+1}, ... can be expressed as a linear combination of $A^0 = I$, A, ..., A^{n-1}
  characteristic polynomial of A:

$$p(\lambda) = \det(\lambda I - A) = a_0 + a_1\lambda + a_2\lambda^2 + \cdots + a_n\lambda^n$$

Then: $a_0 I + a_1 A + a_2 A^2 + \ldots a_n A^n = 0$

$$A^n = \sum_{i=0}^{n-1} \bar{a}_i A^i$$

$$A^{n+1} = A \cdot A^n = \bar{a}_0 A + \bar{a}_1 A^2 + \cdots + \bar{a}_{n-1} A^n =$$

$$= \bar{a}_0 A + \bar{a}_1 A^2 + \cdots + \bar{a}_{n-1} \sum_{i=0}^{n-1} \bar{a}_i A^i$$

and so on $\Longrightarrow$ $A^i B$ and CA^i for $i \geq n$ cannot increase the rank of the controllability and observability matrix, respectively

Geometrical interpretation of observability

(A, C) **unobservability subspace of the system**:

set of initial condition values, which cannot be distinguished from each other knowing (measuring) the output signal

namely, starting the system operation from any initial condition from the unobservability subspace, the system will produce the *same* output

Computing the basis of the unobservability subspace $\ker(\mathcal{O}_n)$

Example

Matrices of the state space model:

$$A = \begin{bmatrix} 1 & 2 \\ -2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad D = 0$$

Observability matrix:

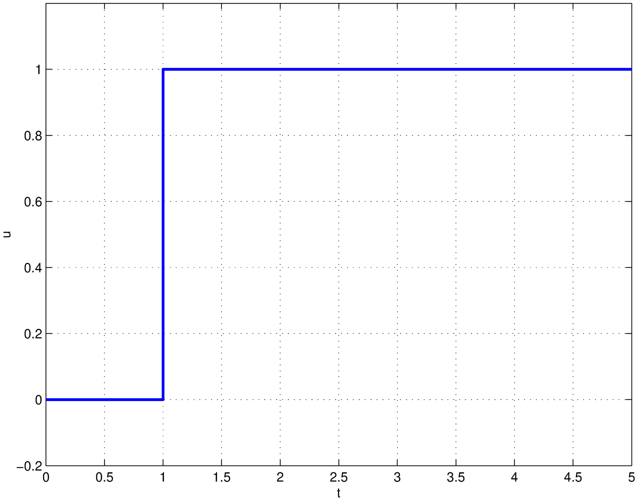
$$\mathcal{O}_2 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

Basis of the unobservability subspace:

$$\ker(\mathcal{O}_2) = \text{span}\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$$

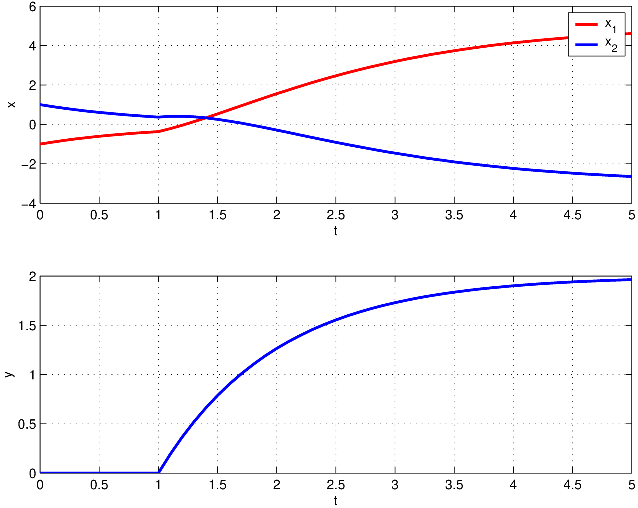
Example

Input given to the system:



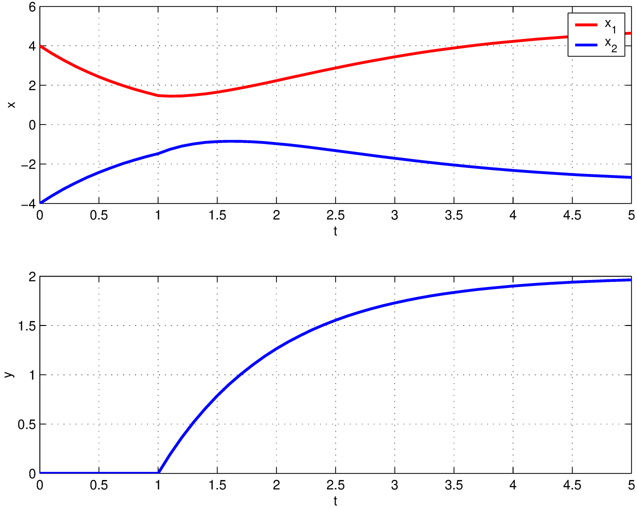
Example

State variables of the system and its output for $x(0) = [-1 \;\; 1]^T$



Example

State variables of the system and its output for $x(0) = [4 \quad -4]^T$



Geometrical interpretation of controllability

(A, B) **controllability subspace of a system**:

set of state vectors $x_1 \in \mathbb{R}^n$, which can be reached in finite time from the origin of the state space $(x(0) = 0)$.
$\exists u : [0, T] \to \mathbb{R}^m, \quad T < \infty$ such that $x(T) = x_1$

in other words, there does not exist any input signal $u(t)$ for which the state vector can leave the controllability subspace.

Computing the basis of the controllability subspace $\mathrm{im}(\mathcal{C}_n)$

Example

Matrices of the state space model:

$$A = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad D = 0$$

Controllability matrix:

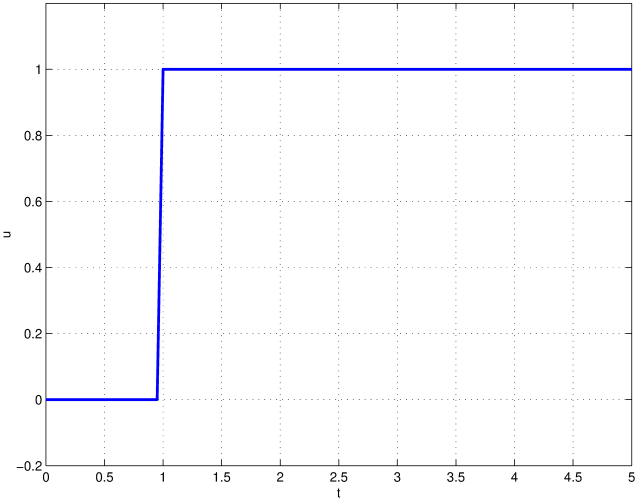
$$\mathcal{C}_2 = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

Basis of the controllability subspace:

$$\mathrm{im}(\mathcal{C}_2) = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix} \right\} = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

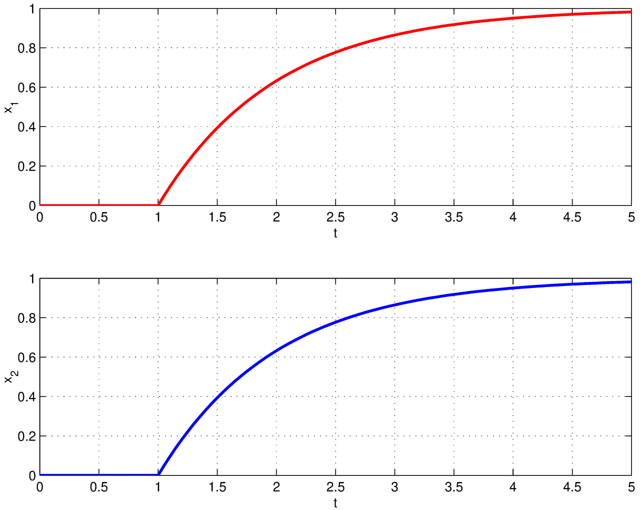
Example

Input given to the system:



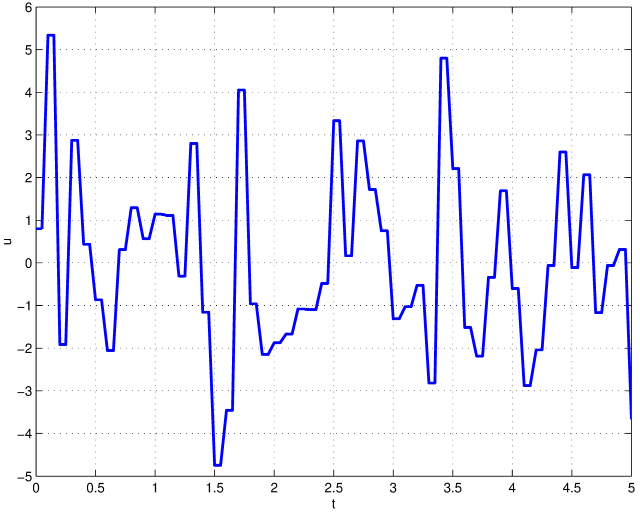
Example

State variables of the system, in case of $x(0) = [0 \;\; 0]^T$



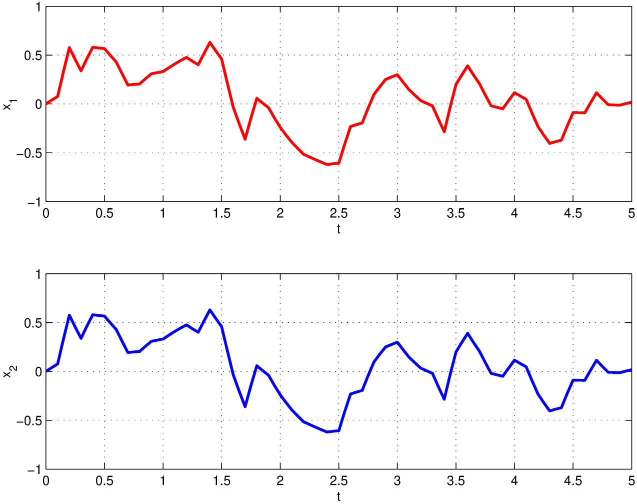
Example

Input given to the system:



Example

State variables of the system, in case of $x(0) = [0 \;\; 0]^T$



Summary

- observability : possibility to compute the state (initial condition) from inputs and outputs knowing the model
- controllability : possibility to reach any target state from any initial condition with appropriate input knowing the model
- necessary and sufficient condition: full rank of the observability/controllability matrix
- geometry: controllability subspace, unobservability subspace