Computer Controlled Systems (Introduction to systems and control theory) Lecture 2

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PPKE-ITK, 19 September, 2019

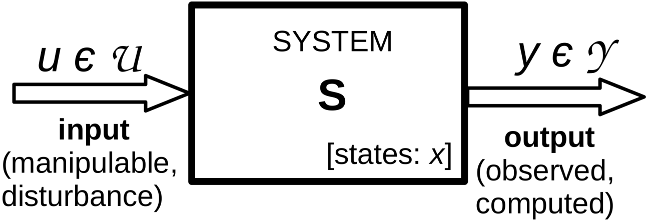
Contents

Systems

System (**S**): performs operations on signals (abstract operator)

$$y = \mathbf{S}[u]$$

- input signal space: $\mathcal{U}$
- output signal space: $\mathcal{Y}$
- inputs: $u \in \mathcal{U}$
- output: $y \in \mathcal{Y}$



Systems – example

From the previous lecture: systems with possible inputs and outputs

- RLC circuit, eq.
  - input: u_{be}, output: u_C
  - input: u_{be}, output: i
- Primary circuit pressure control tank
  - input: *heating power*, output: *primary circuit pressure*
- steered car model
  - input: (u_ϕ, u_t), output: (x, y, θ)

1. Systems

2. **Basic system properties**

3. Mathematical models of CT-LTI systems
   - Input output models
   - State space systems

Linearity

$$\mathbf{S}[c_1 u_1 + c_2 u_2] = c_1 y_1 + c_2 y_2 \tag{1}$$

$c_1, c_2 \in \mathbb{R}$, $u_1, u_2 \in \mathcal{U}$, $y_1, y_2 \in \mathcal{Y}$, and
$\mathbf{S}[u_1] = y_1 \quad , \quad \mathbf{S}[u_2] = y_2$
i.e. satisfies the principle of *superposition*

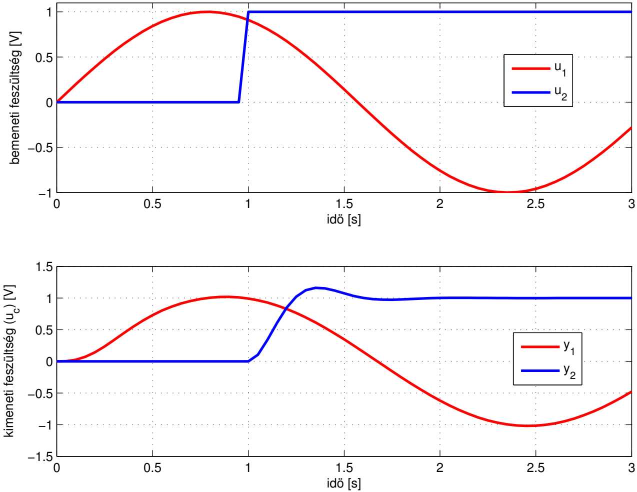
Examples

- the RLC circuit is linear
- the bioreactor model is nonlinear

Checking whether a system is linear or not: by definition (1)

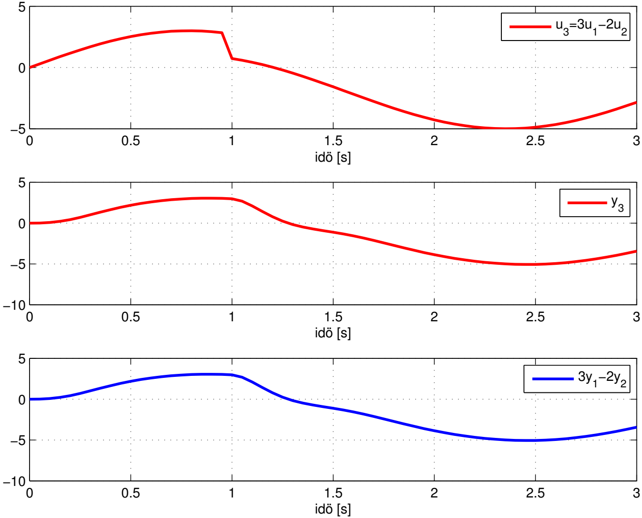
Example: RLC circuit

The system's output for two different inputs:



Example: RLC circuit

The system's output for a linear combination of the previous two inputs:

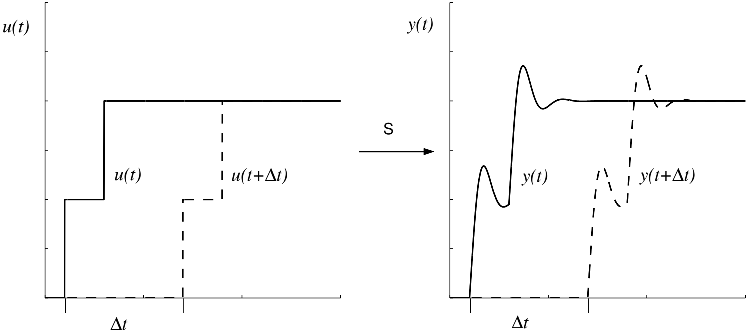


Basic system properties – 2

time invariance: the shift operator and the system operator commute, i.e.

$$\mathbf{T}_\tau \circ \mathbf{S} = \mathbf{S} \circ \mathbf{T}_\tau$$

where $\mathbf{T}_\tau$ denotes the shift operator (in time), i.e. $\mathbf{T}_\tau\, x(t) = x(t-\tau)$
Checking whether a system is time invariant: **constant (time independent) parameters in the system's ordinary differential equations**



Basic system properties – 3

- *continuous time and discrete time systems*
  continuous time: $(\mathcal{T} \subseteq \mathbb{R})$
  discrete time: $\mathcal{T} = \{\cdots, t_0, t_1, t_2, \cdots\}$
- *single input – single output (SISO)*
  multiple input – multiple output (MIMO) systems
- *causal/non causal systems*

CT-LTI system models

- **input-output models of SISO systems**
  - time domain (t)
  - operator domain (s - Laplace transform)
  - frequency domain (ω - Fourier transform)
- **State space models**

CT-LTI system models – 1

Time domain

Linear differential equations with constant coefficients

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_1 \frac{dy}{dt} + a_0 y = b_0 u + b_1 \frac{du}{dt} + \ldots + b_m \frac{d^m u}{dt^m}$$

with given initial conditions

$$y(0) = y_{00} \ , \ \frac{dy}{dt}(0) = y_{10} \ , \ \ldots \ , \ \frac{d^{n-1} y}{dt^{n-1}}(0) = y_{(n-1)0}$$

CT-LTI system models – 2

Operator domain, SISO systems
Transfer function

$$Y(s) = H(s)U(s)$$

if zero initial conditions assumed (!)

$Y(s)$	Laplace transform of the output signal
$U(s)$	Laplace transform of the input signal
$H(s) = \frac{b(s)}{a(s)}$	*the system's transfer function*
	where $a(s)$ and $b(s)$ are polynomials
	$\deg\ b(s) = m$
	$\deg\ a(s) = n$

Strictly proper transfer function: $m < n$
Proper: $m = n$,
improper: $m > n$

CT-LTI system models – 3

Time domain – *Impulse response function*
$Y(s) = H(s)U(s) \rightarrow \mathcal{L}^{-1} \rightarrow y(t) = (h * u)(t)$, i.e.

$$y(t) = \int_0^t h(t-\tau)u(\tau)d\tau = \int_0^t h(\tau)u(t-\tau)d\tau$$

using the definition of Dirac-δ, one can obtain:

$$\int_0^\infty \delta(t-\tau)h(\tau)d\tau = \int_0^t \delta(t-\tau)h(\tau)d\tau = h(t)$$

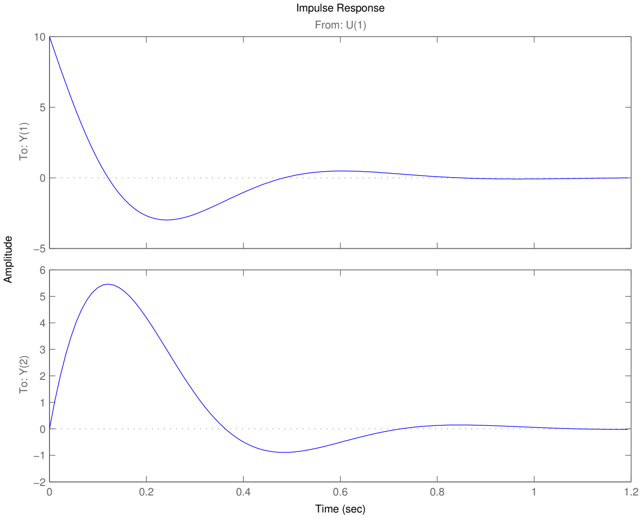
and

$$L(\delta)(s) = \int_0^\infty \delta(t)e^{-st}dt = 1$$

consequently, h is the system's response to a Dirac-δ input

Example

Impulse response functions of the RLC circuit ($u = u_{be}$, $y_1 = i$, $y_2 = u_C$)



CT-LTI I/O models (SISO)

Transfer function – linear differential equation

$$\mathcal{L}\{a_n\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \dots + a_1\frac{dy}{dt} + a_0 y\} =$$
$$= \mathcal{L}\{b_0 u + b_1\frac{du}{dt} + \dots + b_m\frac{d^m u}{dt^m}\}$$

$$H(s) = \frac{Y(s)}{U(s)} = \frac{b(s)}{a(s)}$$

Transfer function – impulse response

$$H(s) = \mathcal{L}\{h(t)\}$$

CT-LTI I/O models: key points

- the Laplace transform converts (higher order) linear differential equations into algebraic equations
- zero initial conditions are assumed for transfer functions (initial state information is not included!)
- knowing the input, the output can be computed (Laplace transform (and inverse), convolution)
- the whole system operator is represented as a time-domain signal ($h(t)$) and/or its Laplace transform ($H(s)$)
- the model parameters are the coefficients in $b(s)$ and $a(s)$

CT-LTI state space systems

General form

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad \text{(state equation)}$$
$$y(t) = Cx(t) + Du(t) \qquad \text{(output equation)}$$

- for a given initial condition $x(t_0) = x(0)$ and $x(t) \in \mathbb{R}^n$,
- $y(t) \in \mathbb{R}^p$, $u(t) \in \mathbb{R}^r$
- model parameters

$$A \in \mathbb{R}^{n \times n}\ ,\ B \in \mathbb{R}^{n \times r}\ ,\ C \in \mathbb{R}^{p \times n}\ ,\ D \in \mathbb{R}^{p \times r}$$

State transformation

$$\dot{x}(t) = Ax(t) + Bu(t) \quad , \quad \dot{\overline{x}}(t) = \overline{A}\overline{x}(t) + \overline{B}u(t)$$
$$y(t) = Cx(t) + Du(t) \quad , \quad y(t) = \overline{C}\overline{x}(t) + \overline{D}u(t)$$

invertible transformation of the states:

$$T \in \mathbb{R}^{n \times n} \quad , \quad det\ T \neq 0 \quad , \quad \overline{x} = Tx \quad \Rightarrow \quad x = T^{-1}\overline{x}$$

$$dim\ \mathcal{X} = dim\ \overline{\mathcal{X}} = n$$

$$T^{-1}\dot{\overline{x}} = AT^{-1}\overline{x} + Bu$$

$$\dot{\overline{x}} = TAT^{-1}\overline{x} + TBu \quad , \quad y = CT^{-1}\overline{x} + Du$$

$$\overline{A} = TAT^{-1} \quad , \quad \overline{B} = TB \quad , \quad \overline{C} = CT^{-1} \quad , \quad \overline{D} = D$$

Transfer function computed from the state space model

Laplace transform of the state space model

$$
\begin{aligned}
sX(s) &= AX(s) + BU(s) \qquad && \text{(state equation, } x(0) = 0\text{)} \\
Y(s) &= CX(s) + DU(s) \qquad && \text{(output equation)}
\end{aligned}
$$

$$
\begin{aligned}
X(s) &= (sI - A)^{-1} BU(s) \\
Y(s) &= \{C(sI - A)^{-1} B + D\} U(s)
\end{aligned}
$$

The system's transfer function $H(s)$, expressed with the corresponding state space model matrices (A, B, C, D):

$$H(s) = C(sI - A)^{-1} B + D$$

Solution of the state space model

We determine the inverse Laplace transform of

$$X(s) = (sI - A)^{-1} BU(s)$$

by considering the Taylor series of (matrix) expression: $(sI - A)^{-1}$:

$$(sI - A)^{-1} = \frac{1}{s}\left(I - \frac{A}{s}\right)^{-1} = \frac{1}{s}\left(I + \frac{A}{s} + \frac{A^2}{s^2} + \dots\right)$$

Thus, the inverse Laplace transform of $(sI - A)^{-1}$ is

$$\mathcal{L}^{-1}\{(sI - A)^{-1}\} = I + At + \frac{1}{2!}A^2 t^2 + \dots = e^{At} \quad , \quad t \geq 0$$

Finally, we obtain the unique solution $x(t)$ of the state space model for the initial condition $x(0)$:

$$\begin{aligned} x(t) &= e^{At}x(0) + \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

Markov parameters

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$
$$y(t) = Cx(t) + Du(t)$$

Assuming $x(0) = 0$, $D = 0$ and $u(t) = \delta(t)$, we obtain the impulse response:

$$h(t) = Ce^{At}B = CB + CABt + CA^2B\frac{t^2}{2!} + ...$$

Markov parameters

$$CA^iB \quad , \quad i = 0, 1, 2, ...$$

are ***invariant*** *for the state transformations.*

state space models: key points

- the Laplace transform converts sets of first order linear differential equations into algebraic equations
- SS models can handle non-zero initial conditions
- knowing the input and the initial condition, the output can be computed (Laplace transform (and inverse), convolution)
- the model parameters are the A, B, C, D matrices ($x(0)$ is also needed for the solution)
- SS models can be easily transformed to I/O models through Laplace transform assuming $x(0) = 0$

Summary

- fundamental system properties: linearity (superposition), time-invariance
- LTI I/O models: higher order linear differential equations containing only the input and the output (and derivatives)
- transfer function, impulse response function: LTI system operators given in the form of signals
- state space models: sets of first order ODEs with state variables, inputs and outputs ; initial conditions not necessarily zero
- SS and I/O models can be converted to each other
- key role of Laplace transform in handling/solving I/O and SS models