Computer Controlled Systems

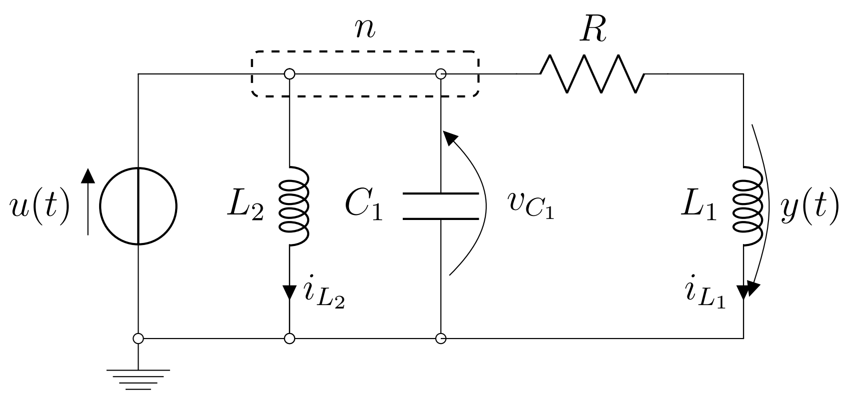
Homework 2.

Submission deadline: 5th of November, at 10:00 (approx. 4 weeks)

All solutions are expected to be calculated by hand, also all figures have to be drawn by hand. Computer programs (e.g. Matlab) can be used for self-verification, but all problems have to contain the detailed steps of solutions

Problems

1. Let us consider the following circuit:



The input of the system is the $u(t)$ current and the output of the system is the $y(t)$ voltage drop across the L_1 inductor.
For the derivation of a state space representation we want to use as few state variables as possible. There are three energy storage elements, so our goal is to use three state variables. We will choose i_{L_1}, i_{L_2} and v_{C_1}, i.e. the currents across the inductors and the voltage drop across the capacitor. We know that

$$L_2 \frac{\mathrm{d}i_{L_2}}{\mathrm{d}t} = v_{C_1} \implies \frac{\mathrm{d}i_{L_2}}{\mathrm{d}t} = \frac{1}{L_2} v_{C_1}.$$

Also using Kirchoff's first law at the junction n we get:

$$u(t) - i_{L_1} - i_{L_2} - C_1 \frac{\mathrm{d}v_{C_1}}{\mathrm{d}t} = 0 \implies \frac{\mathrm{d}v_{C_1}}{\mathrm{d}t} = \frac{1}{C_1}\big(u(t) - i_{L_1} - i_{L_2}\big)$$

We will derive the third equation from the output equation, i.e. we need to formulate that first. Using Kirchoff's second law, the voltage drop across L_1 is

$$y(t) = L_1 \frac{\mathrm{d}i_{L_1}}{\mathrm{d}t} = v_{C_1} - R i_{L_1},$$

which is considered to be the measured output of the system. The third equation of the dynamics is:

$$\frac{\mathrm{d}i_{L_1}}{\mathrm{d}t} = \frac{1}{L_1}(v_{C_1} - R i_{L_1})$$

Denoting our state variables as $x_1 = i_{L_2}$, $x_2 = v_{C_1}$ and $x_3 = i_{L_1}$ we get the following state space representation:

$$\begin{aligned}
\dot{x}_1 &= \frac{1}{L_2}x_2 \\
\dot{x}_2 &= -\frac{1}{C_1}(x_1 + x_3) + \frac{1}{C_1}u \\
\dot{x}_3 &= \frac{1}{L_1}x_2 - \frac{R}{L_1}x_3 \\
y &= x_2 - Rx_3.
\end{aligned}$$

(a) Determine the matrices A, B, C, D and rewrite the state space representation using the matrices!

(b) For the following exercises consider the following parameters for the circuit:

$$C_1 = 10mF \quad L_1 = 10H \quad L_2 = 100H \quad R = 60\Omega.$$

(c) Give the transfer function $H(s)$ of the system!

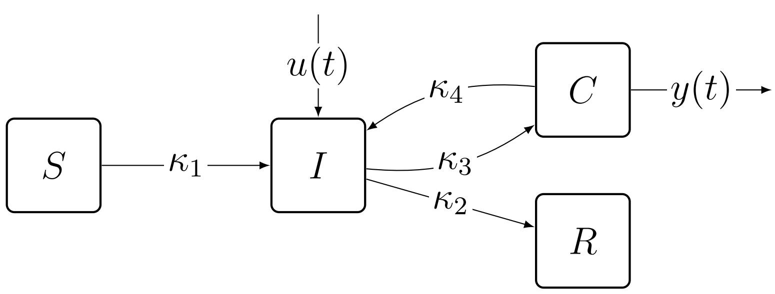
(d) Determine the eigenvalues of matrix A! Is the system stable?

(e) The energy stored in a capacitor and an inductor is

$$\begin{aligned}
E_C &= \frac{1}{2}Cv_C^2 \\
E_L &= \frac{1}{2}Li_L^2.
\end{aligned}$$

Check that the total energy function $V = E_{C_1} + E_{L_1} + E_{L_2}$ is an appropriate Lyapunov function.

2. The SIR model is one of the simplest models used in epidemiology, consisting of three compartments, namely S is the group of susceptible people, I is the group of infectious people and R is the group of recovered (immune) people. The variables $x_1(t)$, $x_2(t)$ and $x_3(t)$ describe the number of people in the S, I, and R groups, respectively.
There are many variations of this model and we will also consider an additional compartment. Some people with infectious diseases never completely revocer and continue to carry the infection, without sufering from the symptoms. Thus the additional compartment will be the group of the carriers, which will be denoted by C, and the number of people in the group will be denoted by $x_4(t)$.
Consider also a vaccination, which will be the input in our system. Unfortunately these vaccinations can't prevent our disease, it is only a treatment. Furthermore assume that we can effectively measure the carriers, i.e. the number of the carriers is the output of the system.
We can draw the following diagram to visualize the dynamics:



The following equations describe the dynamics of the system:

$$\begin{aligned}
\dot{x}_1 &= -\kappa_1 x_1 \\
\dot{x}_2 &= \kappa_1 x_1 + \kappa_4 x_4 - (\kappa_2 + \kappa_3)x_2 + u \\
\dot{x}_3 &= \kappa_2 x_2 \\
\dot{x}_4 &= \kappa_3 x_2 - \kappa_4 x_4 \\
\dot{y} &= x_4
\end{aligned}$$

(a) Rewrite the state space representation using matrices!

(b) Determine the controllability and observability matrices of the system!
Hint. The values of parameters k_i are intentionally suppressed. The exact parametric value of $\mathcal{O}_n$ and $\mathcal{C}_n$ can be computed using Matlab's Symbolic Math Toolbox. But, try to concentrate only on the zeros of matrices A^k, $\mathcal{C}_n$ and $\mathcal{O}_n$. E.g.

$$A^2 = \begin{pmatrix} \star & 0 & 0 & 0 \\ \star & \star & 0 & \star \\ .. & & & \\ .. & & & \end{pmatrix} \tag{1}$$

(c) Compute the controllable subspace and the unobservable subspace of the system!

(d) Discuss the physical interpretation of the obtained subspaces!

Compulsory only for TP students, ***but extra points for others***.

(e) Give a minimal realization of the system without computing the transfer function!