

Computer controlled systems

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Pole-placement controller

Pole-placement controller based on Bass-Gura formula

$$K = (\underline{\alpha} - \underline{a})T_l^{-1}C^{-1}$$

where $\underline{\alpha}$ is the expected (prescribed) characteristic polynomial of the closed-loop system, \underline{a} is the characteristic polynomial of the original (uncontrolled) system, C is the controllability matrix, finally T_l is the following Toeplitz matrix:

$$T_l = \begin{pmatrix} 1 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & 1 & a_1 & \cdots & a_{n-2} \\ 0 & 0 & 1 & \cdots & a_{n-3} \\ \cdot & \cdot & \cdot & \cdots & \cdot \end{pmatrix}$$

Example 1. Design a pole-placement controller for the following CT LTI SISO system:

$$A = \begin{pmatrix} 2 & -2 \\ 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad C = (1 \quad 1)$$

Solution.

$$a(s) = s^2 - 3s + 2$$

$$a_1 = -3$$

$$a_2 = 2$$

The prescribed characteristic polynomial ($\phi_c(s)$):

$$\alpha(s) = s^2 + 3s + 2$$

$$\alpha_1 = 3$$

$$\alpha_2 = 2$$

A Toeplitz matrix and the controllability matrix in this case are

$$T_l = \begin{pmatrix} 1 & a_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix} \quad C = \begin{pmatrix} 1 & -2 \\ 2 & 2 \end{pmatrix}$$

$$T_l^{-1} = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \quad C^{-1} = \frac{1}{6} \begin{pmatrix} 2 & 2 \\ -2 & 1 \end{pmatrix}$$

Then the static state feedback will be the following:

$$K = (3 - (-3) \quad 2 - 2) \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \frac{1}{6} \begin{pmatrix} 2 & 2 \\ -2 & 1 \end{pmatrix} = (-4 \quad 5)$$

Ackermann formula

$$K = [0 \ 0 \ \dots \ 0 \ 1] \mathcal{C}_n^{-1} \phi_c(A)$$

where $\phi_c(s)$ is the prescribed characteristic polynomial of the closed-loop (controlled) system. In the previous example, it was denoted by $\alpha(s) = \phi_c(s)$.

Example 2. Design a pole-placement controller for the following CT LTI SISO system:

$$A = \begin{pmatrix} 2 & -2 \\ 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad C = (1 \ 1)$$

Solution.

$$\mathcal{C}_2 = (B \ AB) = \begin{pmatrix} 1 & -2 \\ 2 & 2 \end{pmatrix} \rightarrow \mathcal{C}_2^{-1} = \begin{pmatrix} \frac{1}{3} & \frac{1}{6} \\ -\frac{1}{3} & \frac{1}{6} \end{pmatrix}$$

Legyen $\lambda_1 = -1$ és $\lambda_2 = -2$.

$$\phi_c = (s - \lambda_1)(s - \lambda_2) = s^2 + 3s + 2$$

$$\phi_c(A) = A^2 + 3A + 2I = \begin{pmatrix} 12 & -12 \\ 0 & 6 \end{pmatrix}$$

$$K = (0 \ 1) \begin{pmatrix} \frac{1}{3} & \frac{1}{6} \\ -\frac{1}{3} & \frac{1}{6} \end{pmatrix} \begin{pmatrix} 12 & -12 \\ 0 & 6 \end{pmatrix} = (-4 \ 5)$$

Check

$$A - BK = \begin{pmatrix} 2 & -2 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix} (-4 \ 5) = \begin{pmatrix} 6 & -7 \\ 8 & -9 \end{pmatrix}$$

$$\det(\lambda I - (A - BK)) = \lambda^2 + 3\lambda + 2$$

Namely, the poles of the obtained closed-loop system are indeed the prescribed values.

Example 3. Design a pole-placement controller for the following CT LTI SISO system:

$$A = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad C = (1 \ 1)$$

Solution.

$$\mathcal{C}_2 = (B \ AB) = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \rightarrow \mathcal{C}_2^{-1} = \begin{pmatrix} 1 & -\frac{2}{3} \\ 0 & \frac{1}{3} \end{pmatrix}$$

Let $\lambda_1 = -1$ and $\lambda_2 = -2$.

$$\phi_c = (s + \lambda_1)(s + \lambda_2) = s^2 + 3s + 2$$

$$\phi_c(A) = A^2 + 3A + 2I = \begin{pmatrix} 9 & -3 \\ 9 & -3 \end{pmatrix}$$

$$K = (0 \ 1) \begin{pmatrix} 1 & -\frac{2}{3} \\ 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 9 & -3 \\ 9 & -3 \end{pmatrix} = (3 \ -1)$$

Check:

$$A - BK = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} (3 \ -1) = \begin{pmatrix} -1 & 0 \\ 3 & -2 \end{pmatrix}$$

$$\det(\lambda I - (A - BK)) = \lambda^2 + 3\lambda + 2$$

Indeed, the poles of the closed loop system are the prescribed values.

Example 4. Given the following CT LTI SISO systems

$$1. \quad \begin{cases} \dot{x} = \begin{pmatrix} 2 & 0 \\ 9 & -3 \end{pmatrix} x + \begin{pmatrix} 0 \\ 3 \end{pmatrix} u \\ y = (1 \quad 1) x \end{cases} \quad \Bigg| \quad 2. \quad \begin{cases} \dot{x} = \begin{pmatrix} 2 & 0 \\ 9 & -3 \end{pmatrix} x + \begin{pmatrix} 2 \\ 3 \end{pmatrix} u \\ y = (1 \quad 1) x \end{cases}$$

Design a state feedback controller (if it is possible), that stabilizes the system!

Example 5. Given the following CT LTI SISO system $H(s) = \frac{2s-4}{s^2+s-6}$.

1. Is the system asymptotically stable?
2. If it is possible, design a controller, that shifts the system's poles to $p_1 = -3$ and $p_2 = -5$! Hint: controllability normal form.

Linear state observer design

Goal: computation of the values of the non-measured state variables of the system using the observed output.

The dynamical system

$$\frac{d\hat{x}}{dt} = F\hat{x} + Ly + Hu$$

is called a full order state observer, if the error dynamics $e = x - \hat{x}$ tends to zero, i.e. $\lim_{t \rightarrow \infty} e = 0$

In case of an LTI system:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

$$\begin{aligned} \dot{e} &= \dot{x} - \dot{\hat{x}} = Ax + Bu - F\hat{x} - Ly - Hu + Fx - Fx = \\ &= Ax + Bu - F\hat{x} - LCx - Hu + Fx - Fx = \\ &= (A - LC - F)x + (B - H)u + F(x - \hat{x}) = (A - LC - F)x + (B - H)u + F(e) \end{aligned}$$

Let $F = A - LC$ and $H = B$

Then $\dot{e} = Fe$

We require that the system be asymptotically stable, namely the real part of the roots of the characteristic polynomial $\det(sI - (A - LC))$ be negative.

$$\det(sI - (A - LC)) = \det(sI - (A^T - C^T L^T))$$

We can observe that the state observer design can be traced back to a pole placement problem of (A', B') , where $A' = A^T$, $B' = C^T$, and the result (K) of the pole placement should be interpreted as $L = K^T$.

Example 6. Design a state observer for the following CT LTI SISO system

$$A = \begin{pmatrix} -3 & 1 \\ 2 & -1 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad C = (0 \ 1)$$

Solution.

Let the characteristic polynomial of the closed-loop system: $\phi_o(s) = (s + 3)(s + 3)$

In order to use the Ackermann, formula we should substitute $A' = A^T$ into $\phi_o(s)$:

$$\phi_o(A') = \begin{pmatrix} 2 & 4 \\ 2 & 6 \end{pmatrix}$$

If $B' = C^T$, the obtained controllability matrix for (A', B') (which is actually the transpose of the observability matrix of (A, C)) is:

$$C'_2 = \begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix}$$

Its inverse will be:

$$(C'_2)^{-1} = \begin{pmatrix} 1/2 & 1 \\ 1/2 & 0 \end{pmatrix}$$

Finally, we compute the feedback gain K :

$$K = (0 \ 1) \begin{pmatrix} 1/2 & 1 \\ 1/2 & 0 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 2 & 6 \end{pmatrix} = (1 \ 2)$$

From this:

$$L = K^T = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad F = A - LC = \begin{pmatrix} -3 & 0 \\ 2 & -3 \end{pmatrix} \quad H = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Example 7. Design a state observer for the following CT LTI SISO system

$$A = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad C = (1 \ 0)$$

Example 8. Design a state observer AND a stabilizer state feedback controller for the following CT LTI SISO system.

$$A = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad C = (1 \ 0)$$

Separation principle: the observer gain L and the feedback gain K can be designed separately.

Optimal state feedback controller - LQR controller design

We want to minimize the following functional:

$$J(x, u) = \frac{1}{2} \int_0^T x^T Q x + u^T R u dt$$

where Q and R are positive definite symmetric matrices. In case of LTI systems this problem can be traced back to a CARE (continuous-time algebraic Riccati equation):

$$KA + A^T K - KBR^{-1}B^T K + Q = 0$$

The system can be stabilized with the $u = -Gx$ state feedback, where

$$G = R^{-1}B^T K$$

Example 9. Design an optimal LQR controller for the following system: $\dot{x} = 2x + u$, i.e $A = 2, B = 1$.

Solution. We minimize the following functional:

$$J = \frac{1}{2} \int 5x^2 + u^2 dt$$

meaning that in our case $Q = 5$ and $R = 1$. In this case (first order system – only one single state variable) the CARE will have the following form:

$$-K^2 + 4K + 5 = 0$$

The solutions for K are 5 and -1 . By definition, we should choose the positive one, otherwise, we obtain a positive feedback.

$$G = 1 \cdot 1 \cdot 5 = 5$$

Finally, the computed state feedback: $u = -5x$.