

Computer Controlled Systems

Lecture 8

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Outline

- 1 Problem statement, full state feedback
- 2 Pole-placement controller design
- 3 Examples for controller design
- 4 Dual problem: state observer design
- 5 State observer examples
- 6 The combination of state observer and pole placement controller

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General problem statement

Given:

- a **SISO LTI** system with matrices (A, B, C) .
The poles depend on A (on $a(s)$).
- prescribed (expected) poles defined by polynomial $\alpha(s)$, such that $\deg a(s) = \deg \alpha(s) = n$

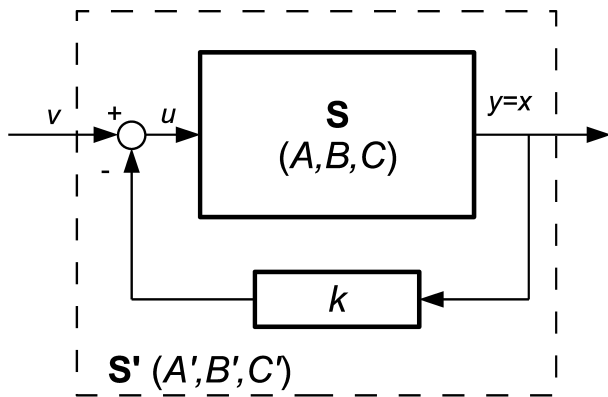
To be computed:

a **full state feedback** such that the poles of the closed loop system will be the roots of $\alpha(s)$.

Subproblem: feedback design, which can stabilize an otherwise unstable system.

Closed loop LTI system – 1

Static linear (full) state feedback:



$$u = -kx + v,$$

where $k \in \mathbb{R}^{r \times n}$, $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^r$

Closed LTI system – 2

The matrices of the SISO system are (A, B, C)

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t)$$

$$y(t), u(t) \in \mathbb{R}, \quad x(t) \in \mathbb{R}^n$$

$$A \in \mathbb{R}^{n \times n}, \quad B \in \mathbb{R}^{n \times 1}, \quad C \in \mathbb{R}^{1 \times n}$$

static linear full state feedback

$$v = u + kx \quad (u = v - kx)$$

$$k = [k_1 \quad k_2 \quad \dots \quad k_n]$$

$$k \in \mathbb{R}^{1 \times n} \quad (\text{row vector})$$

Closed loop system

$$\begin{aligned}\dot{x}(t) &= (A - Bk)x(t) + Bv(t) \\ y(t) &= Cx(t)\end{aligned}$$

Namely:

$$A' = A - B \cdot k, \quad B' = B, \quad C' = C$$

Characteristic polynomials

Without feedback (uncontrolled system):

$$a(s) = \det(sI - A)$$

Closed loop system (controlled) system:

$$a_c(s) = \det(sI - A + Bk)$$

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Determinant of block matrices

Let us calculate the following determinant

$$\det \begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix}$$

in two different (but equivalent) ways

$$\det(M_1) \det(M_4 - M_3 M_1^{-1} M_2) = \det(M_4) \det(M_1 - M_2 M_4^{-1} M_3)$$

We apply:

$$\det \begin{bmatrix} sI - A & B \\ -k & 1 \end{bmatrix}$$

then we obtain the following:

$$\det(sI - A) \det(1 + k(sI - A)^{-1} B) = 1 \cdot \det((sI - A) + B \cdot 1^{-1} \cdot k)$$

Resolvent formula

$$a(s) = s^n + a_1 s^{n-1} + \dots + a_n$$

$$(sI - A)^{-1} = \frac{1}{a(s)} (s^{n-1}I + s^{n-2}(A + a_1I) + s^{n-3}(A^2 + a_1A + a_2I) + \dots)$$

Proof:

$$(sI - A)(sI - A)^{-1} =$$

$$(sI - A) \frac{1}{a(s)} (s^{n-1}I + s^{n-2}(A + a_1I) + s^{n-3}(A^2 + a_1A + a_2I) + \dots) =$$

$$= \frac{1}{a(s)} \left[s^n I \underbrace{-s^{n-1}A + s^{n-1}A}_0 + a_1 s^{n-1} I - s^{n-2} A^2 - s^{n-2} a_1 A + \dots \right] =$$

$$\frac{a(s)}{a(s)} I = I$$

Pole placement – 1

$$\det(sl - A) \cdot \det(1 + k(sl - A)^{-1}B) = 1 \cdot \det((sl - A) + B \cdot 1^{-1} \cdot k)$$

$$a(s)(1 + k(sl - A)^{-1}B) = \det(sl - A + Bk)$$

$$\alpha(s) = a(s)(1 + k(sl - A)^{-1}B) \Rightarrow \alpha(s) - a(s) = a(s)k(sl - A)^{-1}B$$

Using the *resolvent formula*

$$(sl - A)^{-1} = \frac{1}{a(s)}(s^{n-1}I + s^{n-2}(A + a_1I) + s^{n-3}(A^2 + a_1A + a_2I) + \dots)$$

we obtain that

$$\begin{aligned}(\alpha_1 - a_1)s^{n-1} + (\alpha_2 - a_2)s^{n-2} + \dots(\alpha_n - a_n) &= \\ &= kBs^{n-1} + k(A + a_1I)Bs^{n-2} + \dots\end{aligned}$$

Pole placement – 2

$$(\alpha_1 - a_1)s^{n-1} + (\alpha_2 - a_2)s^{n-2} + \dots + (\alpha_n - a_n) = kBs^{n-1} + k(A + a_1I)Bs^{n-2} + \dots$$

polynomial equation

$$\alpha_1 - a_1 = kB$$

$$\alpha_2 - a_2 = kAB + a_1kB = a_1kB + kAB$$

$$\alpha_3 - a_3 = kA^2B + a_1kAB + a_2kB = a_2kB + a_1kAB + kA^2B$$

.

.

$$\underline{\alpha} - \underline{a} = k [B \quad AB \quad A^2B \quad \dots \quad A^{n-1}B] \begin{bmatrix} 1 & a_1 & a_2 & \cdot & \cdot & \cdot & a_{n-1} \\ 0 & 1 & a_1 & \cdot & \cdot & \cdot & a_{n-2} \\ 0 & 0 & 1 & \cdot & \cdot & \cdot & a_{n-3} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

Pole placement controller

$$\underline{\alpha} - \underline{a} = k [B \quad AB \quad A^2B \quad \dots \quad A^{n-1}B] \begin{bmatrix} 1 & a_1 & a_2 & \cdot & \cdot & \cdot & a_{n-1} \\ 0 & 1 & a_1 & \cdot & \cdot & \cdot & a_{n-2} \\ 0 & 0 & 1 & \cdot & \cdot & \cdot & a_{n-3} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

$$\underline{\alpha} - \underline{a} = k C T_\ell^T$$

If S is *controllable* then

$$k = (\underline{\alpha} - \underline{a}) T_\ell^{-T} C^{-1}$$

Controller form realization

$$\begin{aligned}\dot{x}(t) &= A_c x(t) + B_c u(t) \\ y(t) &= C_c x(t)\end{aligned}$$

where

$$A_c = \begin{bmatrix} -a_1 & -a_2 & \cdot & \cdot & \cdot & -a_n \\ 1 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & 1 & 0 \end{bmatrix}, \quad B_c = \begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$$
$$C_c = [b_1 \quad b_2 \quad \cdot \quad \cdot \quad \cdot \quad b_n]$$

The polynomials of the transfer function

$$a(s) = s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n \quad \text{and} \quad b(s) = b_1 s^{n-1} + \dots + b_{n-1} s + b_n$$

$$H(s) = \frac{b(s)}{a(s)}$$

Pole placement controller in case of a controller form

$$A_c - B_c k_c = \begin{bmatrix} -(a_1 + k_{c1}) & -(a_2 + k_{c2}) & \cdot & \cdot & \cdot & -(a_n + k_{cn}) \\ 1 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & 1 & 0 \end{bmatrix}$$

the characteristic polynomial of the closed loop system is $\alpha(s)$:

$$\alpha(s) = \det(sI - (A_c - B_c k_c)) = s^n + (a_1 + k_{c1})s^{n-1} + \dots + (a_n + k_{cn})$$

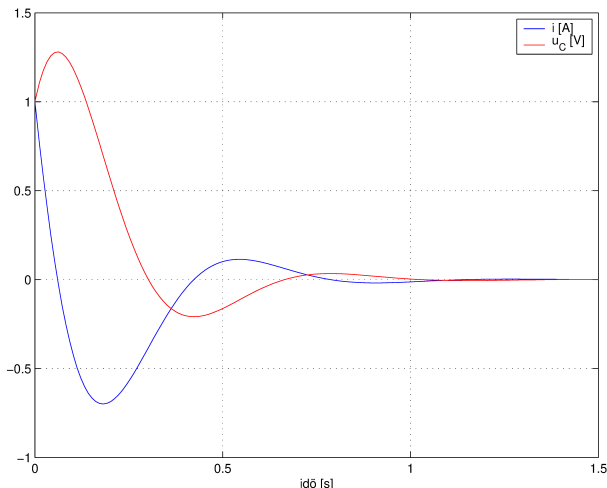
The coefficients k_c of the state feedback gain is

$$k_c = \underline{\alpha} - \underline{a}$$

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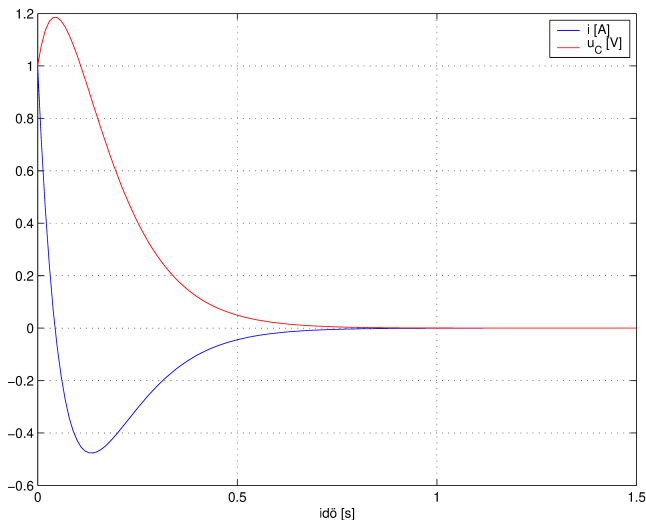
Example – 1

System: RLC circuit. Response of the uncontrolled (open loop) circuit with zero input ($u = 0V$) from initial state $x(0) = [1 \ 1]^T$.
(Poles: $-5 \pm 8.6603j$)



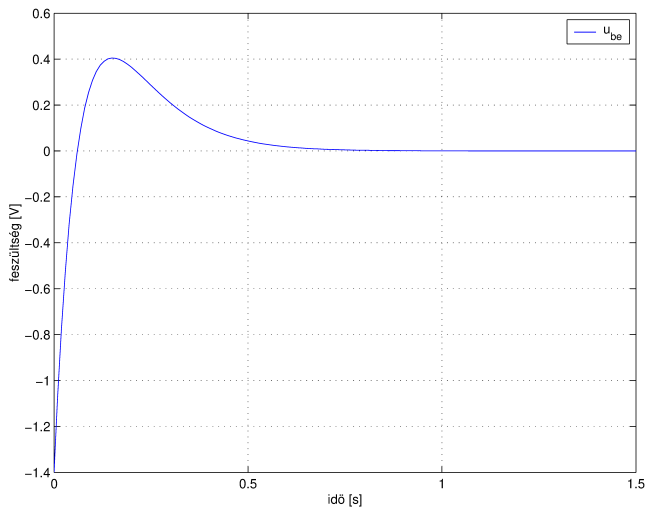
Example – 2

Prescribed poles of the closed loop system: -10 , -12 . Feedback gain: $k = [1.2 \ 0.2]$. Response for $x(0) = [1 \ 1]^T$:



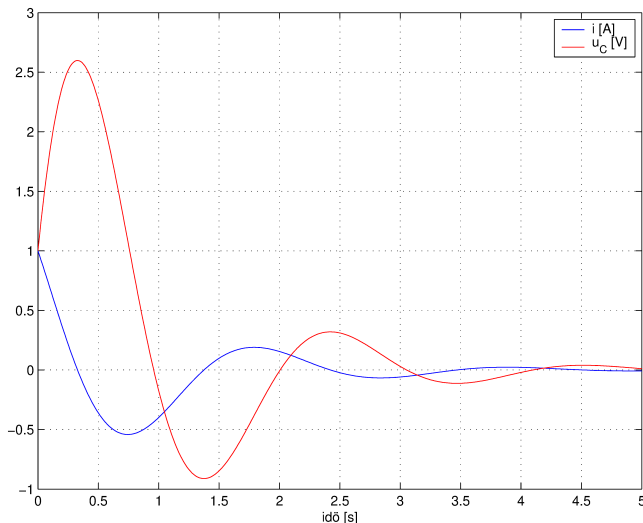
Example – 3

The necessary input for stabilizing control (voltage):



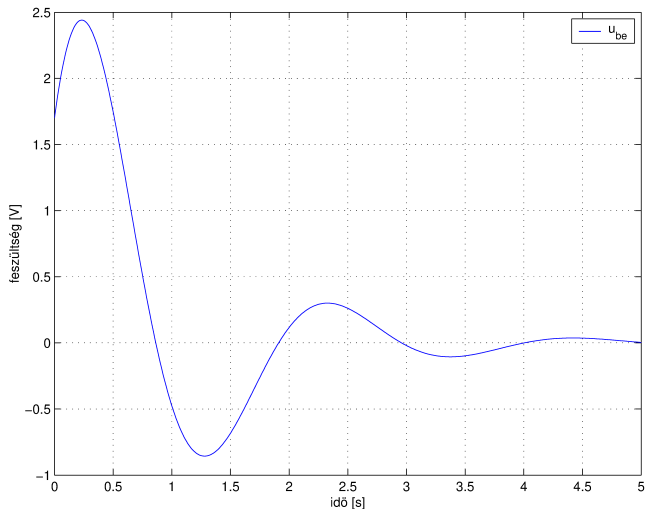
Example – 4

Prescribed poles of the closed loop system: $-1 + 3i$, $-1 - 3i$. Feedback gain: $k = [-0.8 \quad -0.9]$. Response:



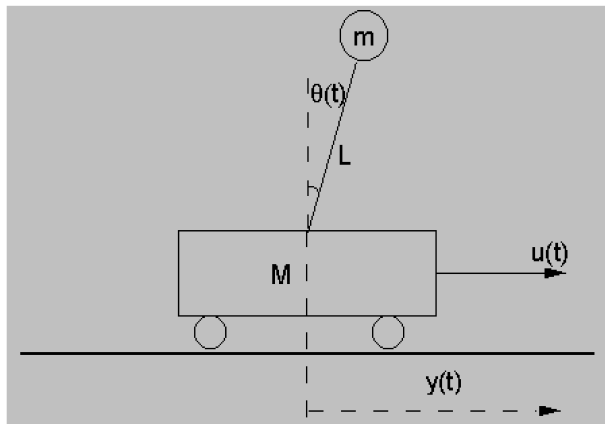
Example – 5

The necessary input for stabilizing control:



Example – 6

System: the inverted pendulum



Example – 7

State vector:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} y \\ \theta \\ \dot{y} \\ \dot{\theta} \end{bmatrix} \quad (1)$$

Equilibrium point: $x^* = [0 \ 0 \ 0 \ 0]^T$

The linearized state-space model:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{mg}{M} & 0 & 0 \\ 0 & \frac{(M+m)g}{ML} & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{M} \\ -\frac{1}{ML} \end{bmatrix}, \quad C = I^{4 \times 4}$$

Parameters: $m = 0.5 \text{ kg}$, $M = 0.1 \text{ kg}$, $L = 1 \text{ m}$, $g = 10 \frac{\text{m}}{\text{s}^2}$

Example – 8

The poles of the uncontrolled system: $\lambda_1=0$, $\lambda_2=0$, $\lambda_3 = 7.746$,
 $\lambda_4 = -7.746$

Goal: stabilizing controller

Prescribed poles of the closed loop system: $\kappa_1 = \kappa_2 = \kappa_3 = \kappa_4 = -1$

The computed feedback gain:

$$k = [-0.01 \quad -6.61 \quad -0.04 \quad -0.44]$$

Example – 9

The operation of the controlled system (simulation: Faludi Gábor)

`ipend_pp-1.avi`

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State observer, problem statement

Recall: If a SSM (A, B, C) is observable, then, knowing the input (u) and the output (y) , the initial state of the system can be computed, and hence every further state values.

Problems:

- The measurement of the input and the output are (in general) not precise enough, furthermore, we need the 1st, 2nd, ..., $(n - 1)$ th derivatives of the output in order to compute the initial condition.
- In general, the system model is not perfect

Goal: design such a tool (state observer), for which we do not need the derivatives of the output y , and the estimated state converges to the actual value of the state vector.

Algebraic form of the state observer

State-space model:

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$

$$\dot{\hat{x}} = (A - LC)\hat{x} + [B \ L] \begin{bmatrix} u \\ y \end{bmatrix}$$

observation error:

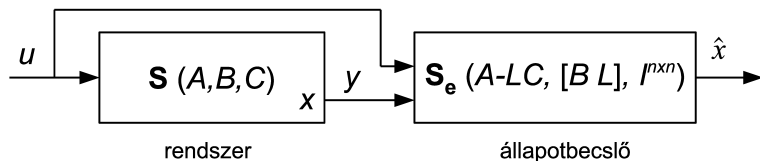
$$e = x - \hat{x}$$

and

$$\dot{e} = (A - LC)e$$

The structure of the state observer

The realization of a state observer (it can be seen from the algebraic equations)



Calculation of the state observer

Reminder: In case of a pole placement controller the system matrices of the closed loop system are $A_c = A - Bk$. (A, B is given, k should be computed, condition: (A, B) is controllable)

System matrix of the state observer: $A_o = A - LC$. (A, C is given, L should be computed, condition: ?)

Solution:

$$A_o^T = A^T - (LC)^T = A^T - C^T L^T$$

In other words, L can be computed using the pole placement algorithm using arbitrary prescribed stable eigenvalues for A_o (i.e. the state observer be stable). Condition: $[C^T \ A^T C^T \ \dots \ (A^{n-1})^T C^T] = \mathcal{O}_n^T$ is a full-rank matrix, namely, the system is observable.

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Example – 1

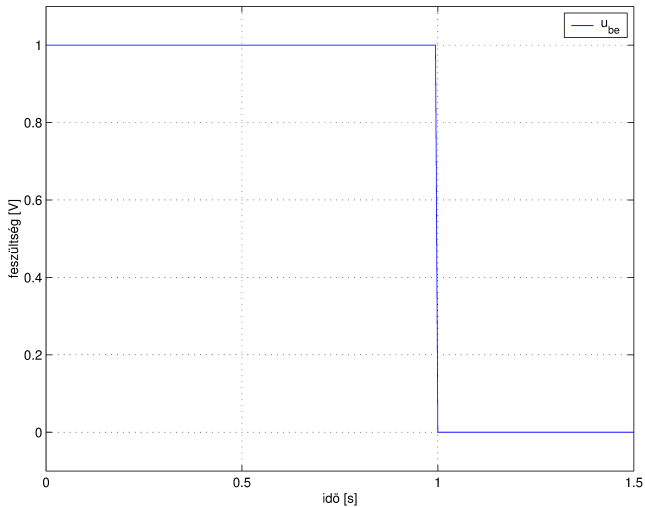
RLC circuit, measured output: u_C , namely $C = [0 \ 1]$

Prescribed eigenvalues of the state observer: $-10, -12$

The computed matrix L of the state observer: $L = [-10 \ 12]^T$

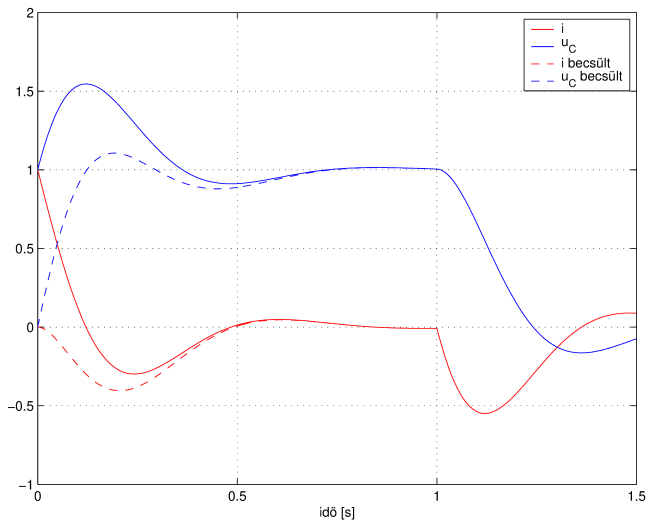
Example – 2

Input of the system:



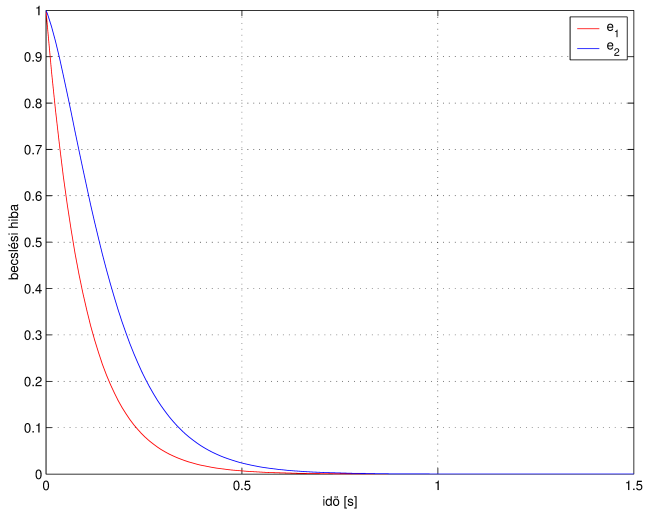
Example – 3

The operation of the state observer:



Example – 4

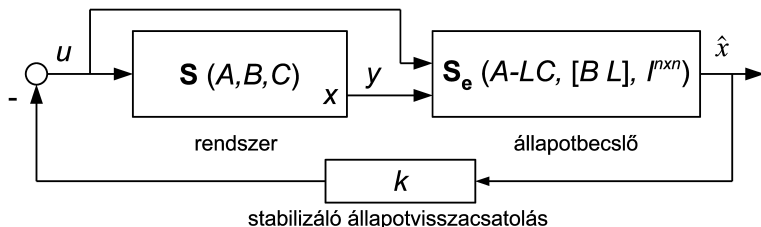
Observation error:



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Separation principle

Problem: what happens if the estimated state is fed back by the computed feedback gain k (dynamic output feedback)?



Separation principle: The stabilizing state feedback with the a stable state observer is asymptotically stable, since the dynamics of the closed loop system is the following:

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \cdot \begin{bmatrix} x \\ e \end{bmatrix}$$

This means that the stabilizing state feedback (K) and a stable state observer (L) can be designed separately.

Separation principle

Computation:

$$\dot{x} = Ax + Bu, \quad u = -K\hat{x}, \quad \text{and: } e = x - \hat{x}$$

From this: $u = -K(x - e) = -Kx + Ke$, and

$$\dot{x} = Ax + B(-Kx + Ke) = (A - BK)x + BKe \quad (2)$$

$$\dot{e} = (A - LC)e \quad (3)$$

Formula for the eigenvalues:

$$\lambda_i \left(\begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \right) = \lambda_j(A - BK) \cup \lambda_k(A - LC),$$

and we know that $A - BK$ ill. $A - LC$ are stability matrices.

Summary

- goal of pole placement: move the poles (eigenvalues) of the controlled system to arbitrary places on the complex plane
- feedback form: full state feedback (requires the knowledge of each state variable)
- condition for computation: controllability
- goal of state observer: asymptotically compute the state variables from the input and the output
- observer gain computation: can be traced back to pole placement (dual problem)
- separation principle: separately designed stabilizing feedback and stable observer results in a stable combined system