Computer Controlled Systems
Lecture 6

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1. Stability criteria for transfer functions
2. SISO systems in the frequency domain
3. Interconnections of subsystems

Transfer functions and stability

SISO case: $H(s) = C(sI - A)^{-1}B = \frac{b(s)}{a(s)} =$

$$\frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} = \frac{(s-\beta_1)(s-\beta_2)\ldots(s-\beta_m)}{(s-\lambda_1)(s-\lambda_2)\ldots(s-\lambda_n)}$$

- Zeros: $\beta_1, \beta_2, \ldots, \beta_m \in \mathbb{C}$
- Poles: $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$ (identical to the eigenvalues of A)

Asymptotic stability $\Leftrightarrow$ $\mathrm{Re}(\lambda_i) < 0$

Routh's stability criterion – 1

$$a(s) = a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n$$

$$\begin{array}{ccccc}
a_0 & a_2 & a_4 & a_6 & \dots \\
a_1 & a_3 & a_5 & a_7 & \dots \\
\frac{a_1 a_2 - a_0 a_3}{a_1} & \frac{a_1 a_4 - a_0 a_5}{a_1} & \frac{a_1 a_6 - a_0 a_7}{a_1} & \dots & \\
\dots & & & & \\
a_n & \dots & & &
\end{array}$$

Routh-coefficients: $R_0 = a_0$, $R_1 = a_1$, $R_2 = \frac{a_1 a_2 - a_0 a_3}{a_1}$, $\dots$, $R_n = a_n$.
(elements of the first column)

Routh's stability criterion – 2

number of sign changes in the column of coefficients = number of roots with positive real part (unstable)
necessary and sufficient condition for stability: $R_i > 0$, $i = 0, \dots, n$.
Example: $a(s) = s^3 + s^2 + 3s + 10$.
$R_0 = 1$, $R_1 = 1$, $R_2 = -7$, $R_3 = 10 \Rightarrow$ 2 roots with positive real parts (unstable system)
Remarks:

- necessary condition for stability (not sufficient for polynomials with degree greater than 2): all coefficients a_i are positive
- in the case of purely imaginary root(s), zero(s) appear among the coefficients

$$W = \begin{bmatrix} a_1 & a_3 & a_5 & \dots & 0 & 0 & 0 \\ a_0 & a_2 & a_4 & a_6 & \dots & 0 & 0 \\ 0 & a_1 & a_3 & a_5 & \dots & & 0 \\ 0 & a_0 & a_2 & a_4 & a_6 & \dots & 0 \\ \dots & & & & & & 0 \\ 0 & 0 & 0 & \dots & a_{n-3} & a_{n-1} & 0 \\ 0 & 0 & 0 & \dots & a_{n-4} & a_{n-2} & a_n \end{bmatrix} \in \mathbb{R}^{n\times n}$$

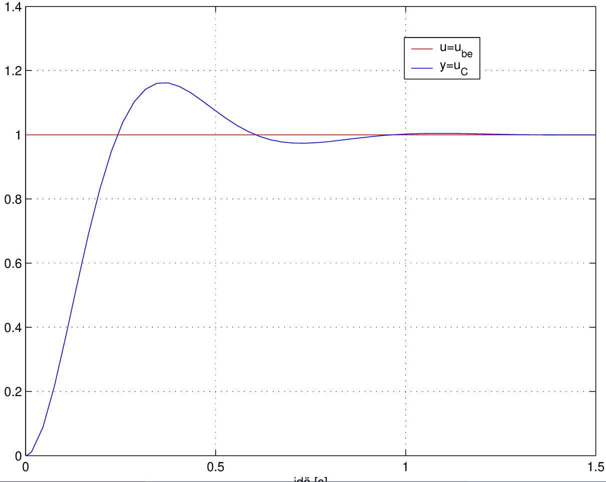
Minors: $H_1, H_2, \dots, H_n$.

Hurwitz's stability criterion – 2

- necessary and sufficient condition for stability: $H_i > 0$, $i = 1, \ldots, n$
- 0 minor: imaginary root
- negative minor: root with positive real part
- relation between Routh- and Hurwitz-coefficients: $R_i = \frac{H_i}{H_{i-1}}$, $H_0 = 1$.

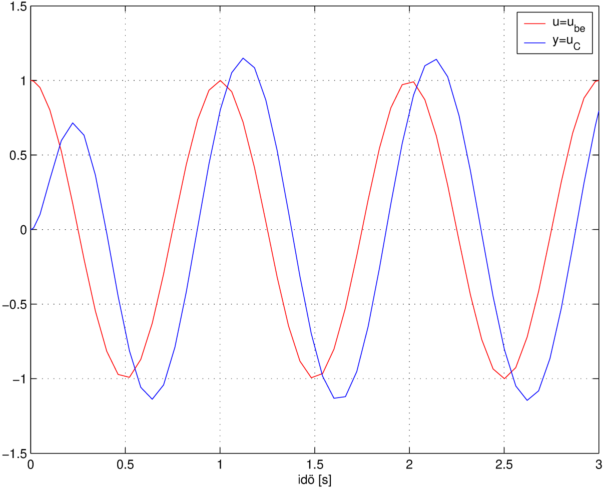
Example: RLC circuit – 1

$R = 1\Omega,\ L = 0.1H,\ C = 0.1F,\ x(0) = [0 \ \ 0]^T,\ y = u_C,$
$u(t) = u_{be} = \cos(\omega \cdot t)$
$\omega = 0 \text{ rad/s} = 0 \text{ Hz}$



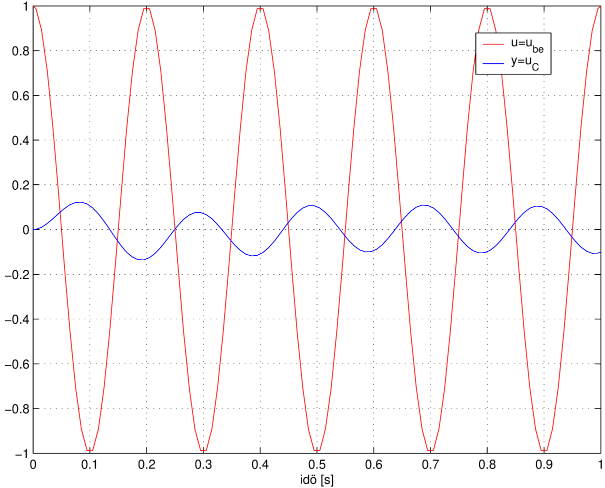
Example: RLC circuit – 2

$\omega = 2\Pi$ rad/s $= 1$ Hz



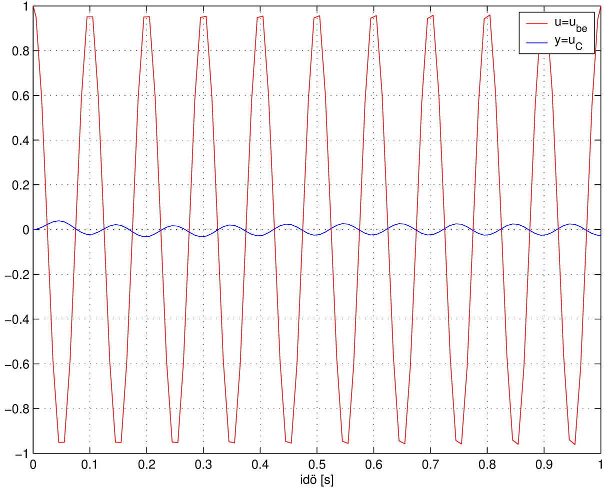
Example: RLC circuit – 3

$\omega = 5 \cdot 2\Pi \text{ rad/s} = 5 \text{ Hz}$



Example: RLC circuit – 4

$\omega = 10 \cdot 2\Pi$ rad/s $= 10$ Hz



Fourier- and Laplace-transforms

Revision: $f: \mathbb{R}_0^+ \mapsto \mathbb{R}$
Fourier-transform:

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt, \quad \omega \in \mathbb{R}$$

Laplace-transform:

$$F(s) = \int_{0}^{\infty} f(t)e^{-st}dt, \quad s \in \mathbb{C}$$

Assume that s is on the imaginary axis. Then: $s \longleftrightarrow j\omega$

Frequency response function

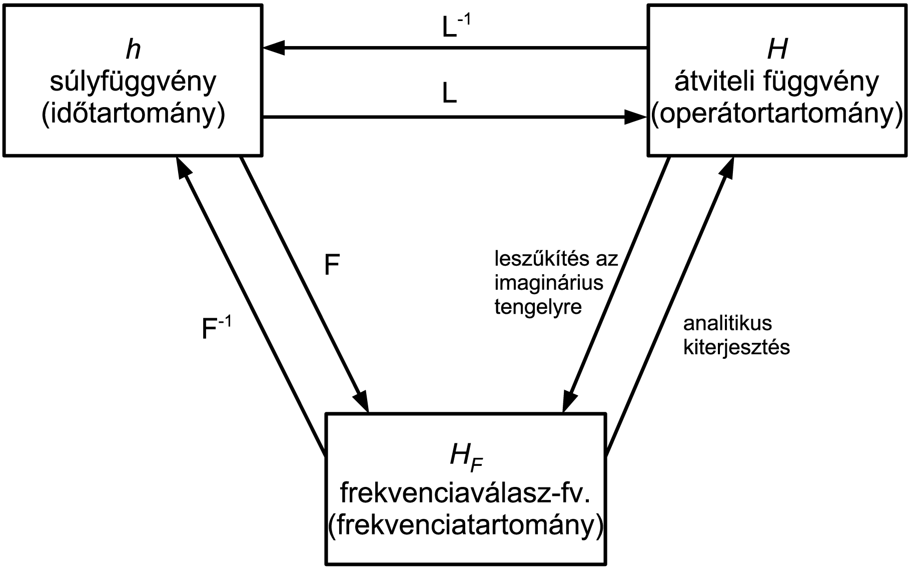
Transfer function: $H(s)$
Definition: $H_F(\omega) = H(j\omega)$ (frequency response function)
Then H_F is the Fourier-transform of the impulse response function (h) since:

$$H_F(\omega) = \int_0^{\infty} h(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty} h(t)e^{-i\omega t}dt$$

H_F is the restriction of H to the imaginary axis
Question: Can we compute H from the restriction on the complex plane, where the Laplace-transform is defined?
Answer: Using the fact that the transfer function is *analytic*, the computation is the following, if the poles of H are on the left half-plane:

$$H(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{H_F(\omega)}{s - i\omega}d\omega$$

Time- operator- and frequency-domains



Response of stable LTI systems to periodic inputs

Theorem: Let $H(s)$ be the transfer function of an asymptotically stable LTI system, and $\omega > 0$. Then the response of the system to the input $u(t) = u_0 \sin(\omega t)$ is of the following form:

$$y(t) = u_0 \mathrm{Re}(H_F(\omega)) \sin(\omega t) + u_0 \mathrm{Im}(H_F(\omega)) \cos(\omega t)$$

(we do not prove)
Remarks:

- It is visible that the output is also periodic with a period $T = \frac{2\pi}{\omega}$ equal to the period of the input.
- The theorem is still valid if the transfer function has purely imaginary poles of the form $i\hat{\omega}$, but $\omega / \hat{\omega} \notin \mathbb{Z}$.

Response of stable LTI systems to periodic inputs

transfer function: $G(j\omega)$, $(G(s))$

$$u(t) = u_0 \sin(\omega t + \alpha)$$

$$y(t) \longrightarrow y_0 \sin(\omega t + \beta)$$

gain: $k = \left|\frac{y_0}{u_0}\right| = |G(j\omega)|$ (frequency dependent!)
phase: $\phi = \beta - \alpha = \angle G(j\omega[\text{rad}])$ (frequency dependent!)
E.g. let $G(j\omega) = a + bj$

$$|G(j\omega)| = \sqrt{(a^2 + b^2)}, \quad \angle G(j\omega) = \arctan(b/a)$$

Gain in time and frequency domains

$$u(t) = a_0 \sin(\omega t), \quad y(t) = a_1 \sin(\omega t + \phi)$$

$$U(s) = \frac{a_0 \omega}{s^2 + \omega^2}, \quad Y(s) = \frac{a_1(s \sin(\phi) + \omega \cos(\phi))}{s^2 + \omega^2}$$

$$|G(j\omega)| = \frac{|Y(j\omega)|}{|U(j\omega)|} = \left| \frac{a_1(j\omega \sin(\phi) + \omega \cos(\phi))}{a_0 \omega} \right| = \left| \frac{a_1}{a_0} \right|$$

$$\angle G(j\omega) = \arctan\left(\frac{\omega \sin(\phi)}{\omega \cos(\phi)} \right) = \phi$$

Example: RLC circuit – 5

Transfer function: $C(sI-A)^{-1}B = \frac{100}{s^2+10s+100} = \frac{100}{(j\omega)^2+10(j\omega)+100}$

- $f = 0$ Hz, $\omega = 0$ rad/s, $G(j\omega) = 1 + 0j$, $|G(j\omega)| = 1$, $\phi = 0$ rad
- $f = 1$ Hz, $\omega = 6.2832$ rad/s, $G(j\omega) = 0.7952 - 0.8256j$, $|G(j\omega)| = 1.1463$, $\phi = -0.8041$ rad
- $f = 5$ Hz, $\omega = 31.4159$ rad/s, $G(j\omega) = -0.1002 - 0.0355j$, $|G(j\omega)| = 0.1063$, $\phi = 0.3404$ rad
- $f = 10$ Hz, $\omega = 62.8319$ rad/s, $G(j\omega) = -0.0253 - 0.004j$, $|G(j\omega)| = 0.0256$, $\phi = 0.1619$ rad

Gain of transfer functions

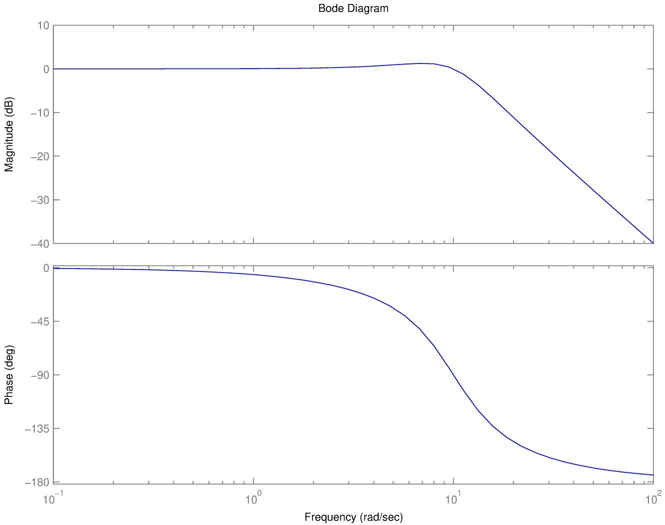
$$A = \left| \frac{y_0}{u_0} \right|$$

in dB:

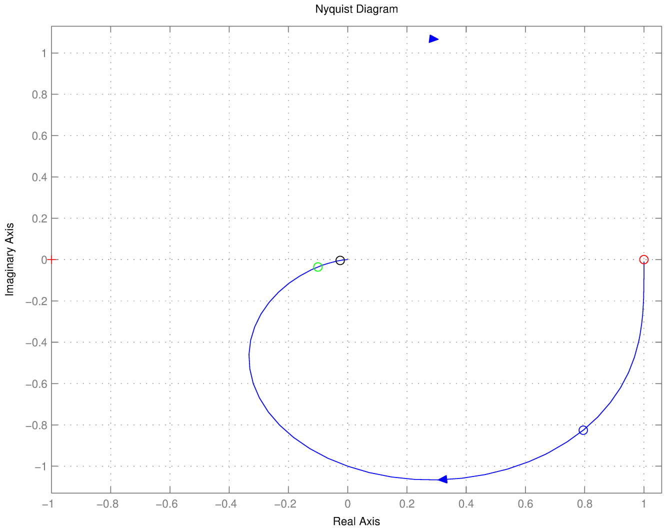
$$A_d = 20 \cdot \log_{10}(A) \text{ [dB]}$$

- $|G(j\omega)| = 1$, $A_d = 0$ dB
- $|G(j\omega)| = 1.1463$, $A_d = 1.1860$ dB
- $|G(j\omega)| = 0.1063$, $A_d = -19.4693$ dB
- $|G(j\omega)| = 0.0256$, $A_d = -31.8352$ dB

Bode-diagram



Nyquist-diagram



Bandwidth of SISO systems

Bandwidth: Frequency, where $|G(j\omega)|$ first crosses the value $1/\sqrt{2}$ (≈ -3 dB) from above

Example: RLC circuit

$y = u_C$, $\omega_c \approx 2.03$ Hz

Transfer function of MIMO systems

$u \in \mathbb{R}^m$, $y \in \mathbb{R}^r$

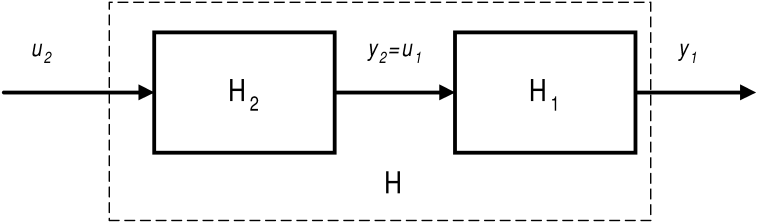
$$Y(s) = H(s)U(s),$$

$$H(s) = \begin{bmatrix} h_{11}(s) & \dots & h_{1m}(s) \\ \\ h_{r1}(s) & \dots & h_{rm}(s) \end{bmatrix} \in \mathbb{C}^{r \times m}$$

Pl. RLC-circuit, $u = u_{in}, y = [i \;\; u_C]^T$

$$H(s) = \begin{bmatrix} \frac{10s}{s^2+10s+100} \\ \frac{100}{s^2+10s+100} \end{bmatrix}$$

Serial interconnection of subsystems

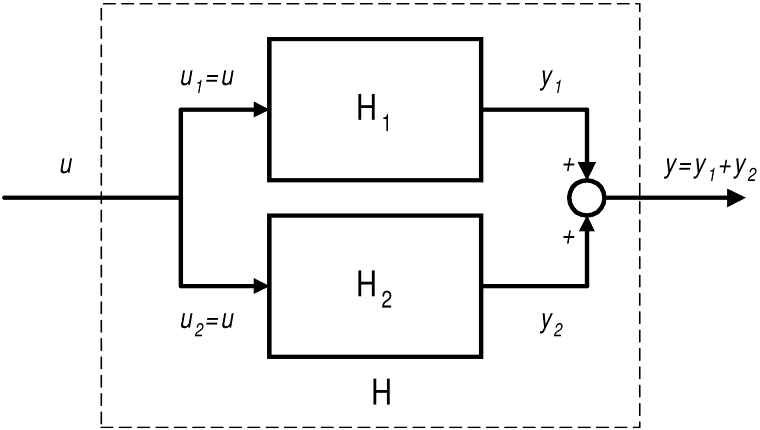


$$H(s) = H_1(s) \cdot H_2(s)$$

i.e.

$$h(t) = (h_1 * h_2)(t)$$

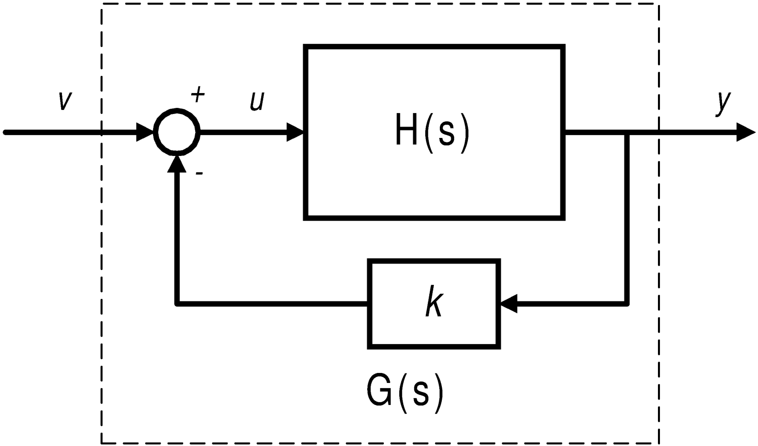
Parallel interconnection of subsystems



$$H(s) = H_1(s) + H_2(s)$$

$$h(t) = h_1(t) + h_2(t)$$

Proportional negative feedback



$$G(s) = \frac{H(s)}{1 + k \cdot H(s)}$$

Negative feedback – example

Original system:

$$H(s) = \frac{1}{s-1}, \quad \text{(unstable)}$$

Feedback system:

$$G(s) = \frac{1}{s+k-1}$$

stable, if $k > 1$

High gain output feedback

$$H(s) = \frac{b(s)}{a(s)}$$

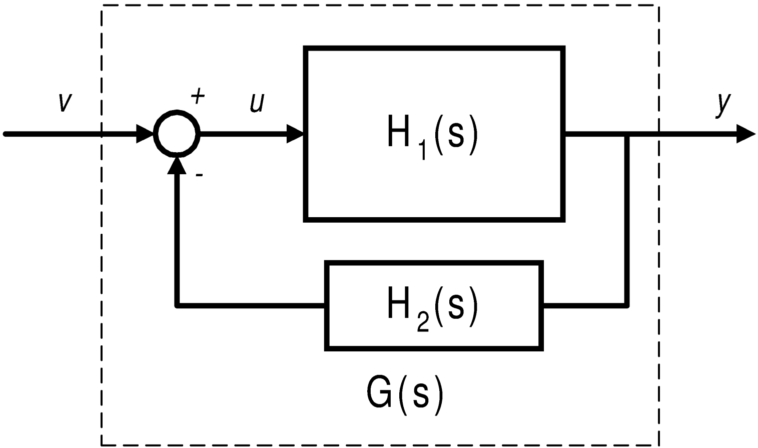
Transfer function of the feedback system:

$$G(s) = \frac{b(s)}{a(s) + k \cdot b(s)} = \frac{n(s)}{d(s)}$$

For $k \to \infty$, $d(s) \to b(s)$, i.e. by increasing the feedback gain, the poles of the feedback system converge to the zeros of the original system.

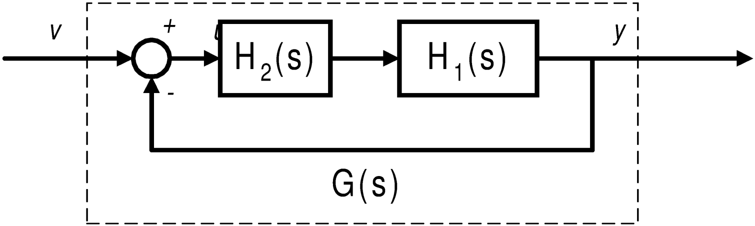
Minimum phase systems: Such systems where the real part of each zero is negative. (They can be stabilized by high gain feedback.)

General negative feedback – 1



$$G(s) = \frac{H_1(s)}{1 + H_1(s)H_2(s)}$$

General negative feedback – 2



$$G(s) = \frac{H_1(s)H_2(s)}{1 + H_1(s)H_2(s)}$$

Summary

- SISO transfer functions (TFs) are complex numbers (with absolute value and angle) at any given s
- frequency domain interpretation: assuming periodic (sinusoidal) input, $s = j\omega$
- absolute value of TF: gain (ratio of O/I amplitudes) at a given frequency
- angle of TF: phase shift at a given frequency
- visualization: Bode diagram, Nyquist diagram
- overall transfer functions were computed for different basic interconnection of subsystems