Computer Controlled Systems
Lecture 5

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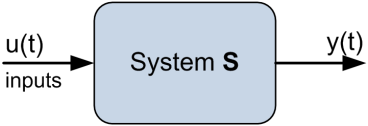
Overview

Systems

- System (**S**): acts on signals

$$y = \mathbf{S}[u]$$

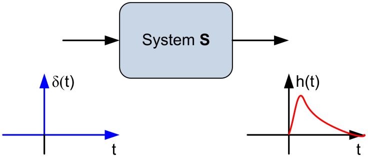
- inputs (u) and outputs (y)



CT-LTI I/O system models

- Time domain: Impulse response function
  is the response of a SISO LTI system to a Dirac-delta input function with zero initial condition.
- The output of **S** can be written as

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)u(\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau$$



CT-LTI state-space models

- **General form - revisited**

$$\dot{x}(t) = Ax(t) + Bu(t) \quad , \quad x(t_0) = x(0)$$
$$y(t) = Cx(t)$$

  with
  - signals: $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^p$, $u(t) \in \mathbb{R}^r$
  - system parameters: $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times r}$, $C \in \mathbb{R}^{p \times n}$ $(D = 0$ by using **centering** the inputs and outputs)
- Dynamic system properties:
  - observability
  - controllability
  - stability

Overview

Signal spaces

- $\mathcal{L}_q$ signal spaces

$$\mathcal{L}_q[0,\infty) = \left\{ f : [0,\infty) \to \mathbb{R} \;\middle|\; f \text{ is measurable and } \int_0^\infty |f(t)|^q \, \mathrm{d}t < 0 \right\}$$

special case

$$\mathcal{L}_\infty[0,\infty) = \left\{ f : [0,\infty) \to \mathbb{R} \;\middle|\; f \text{ is measurable and } \sup_{t\geq 0} |f(t)| < \infty \right\}$$

- Remark: $\mathcal{L}_q$ spaces are Banach spaces with norms

$$\|f\|_q = \left(\int_0^\infty |f(t)|^q \, \mathrm{d}t \right)^{1/q}$$

$$\|f\|_\infty = \sup_{t\geq 0} |f(t)|$$

Vector valued signals

- $\mathcal{L}_q^n$ multidimensional signal spaces

  Let $f(t) \in \mathbb{R}^n, \ \forall t \geq 0$, then

  $$\mathcal{L}_q^n[0,\infty) = \left\{ f : [0,\infty) \to \mathbb{R}^n \ \middle| \ f \text{ is measurable}, \ \int_0^\infty \|f(t)\|_2^q \, \mathrm{d}t < \infty \right\}$$

  where $\|f(t)\| = \sqrt{f^T(t) f(t)}$ is the Euclidean norm in $\mathbb{R}^n$

- $\mathcal{L}_q^n$ is a Banach space equipped with the signal norm

  $$\text{norm:} \ \|f\|_q = \left(\int_0^\infty \|f(t)\|_2^q \, \mathrm{d}t \right)^{1/q}$$

- Remark: The case $\mathcal{L}_2$ is special, because the norm can be originated from an inner product (therefore, $\mathcal{L}_2$ is a Hilbert-space)

BIBO stability – general

Definition (BIBO stability)

A system is *externally or BIBO stable* if for any bounded input it responds with a bounded output

$$\|u\| \leq M_1 < \infty \Rightarrow \|y\| \leq M_2 < \infty$$

where $\|\cdot\|$ is a signal norm.

- This applies to any type of systems.
- Stability is a system property, i.e. it is realization-independent.

BIBO stability – 1

- Bounded input-bounded output (BIBO) stability for SISO systems

$$|u(t)| \leq M_1 < \infty,\ \forall t \geq 0 \quad \Rightarrow \quad |y(t)| \leq M_2 < \infty,\ \forall t \geq 0$$

Theorem (BIBO stability)

A SISO LTI system is BIBO stable if and only if

$$\int_0^\infty |h(t)|\mathrm{d}t \leq M < \infty$$

where $M \in \mathbb{R}^+$ and h is the impulse response function.

BIBO stability – 2

Proof:

$\Leftarrow$ Assume $\int_0^\infty |h(t)|dt \leq M < \infty$ and u is bounded, i.e. $|u(t)| \leq M_1 < \infty, \forall t \in \mathbb{R}_0^+$. Then

$$|y(t)| \leq |\int_0^\infty h(\tau)u(t-\tau)d\tau| \leq M_1 \int_0^\infty |h(\tau)|d\tau \leq M_1 \cdot M = M_2$$

$\Rightarrow$ (indirect) Assume $\int_0^\infty |h(\tau)|d\tau = \infty$, but the system is BIBO stable. Consider the **bounded** input:

$$u(t-\tau) = \text{sign } h(\tau) = \begin{cases} 1 & \text{if} \quad h(\tau) > 0 \\ 0 & \text{if} \quad h(\tau) = 0 \\ -1 & \text{if} \quad h(\tau) < 0 \end{cases}$$

Overview

Stability of nonlinear systems

- Consider the autonomous nonlinear system:

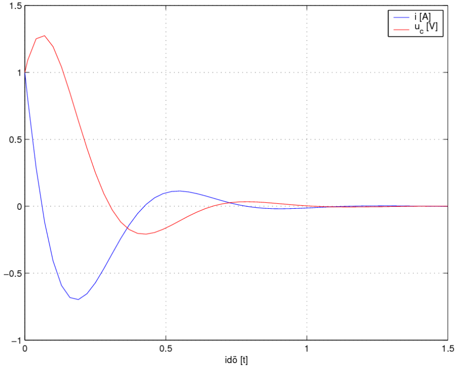
$$\dot{x} = f(x), \quad x \in \mathcal{X} = \mathbb{R}^n, \ f : \mathbb{R}^n \to \mathbb{R}^n$$

with an equilibrium point: $f(x^*) = 0$

  - x^* stable equilibrium point: for any $\varepsilon > 0$ there exists $\delta \in (0, \varepsilon)$ such that for $\|x^* - x(0)\| < \delta$ $\|x^* - x(t)\| < \varepsilon$ holds.
  - x^* asymptotically stable equilibrium pint: x^* stable and $\lim_{t\to\infty} x(t) = x^*$.
  - x^* unstable equilibrium point: not stable
  - x^* locally (asymptotically) stable: there exists a neighborhood U of x^* within which the (asymptotic) stability conditions hold
  - x^* globally (asymptotically) stable: $U = \mathbb{R}^n$

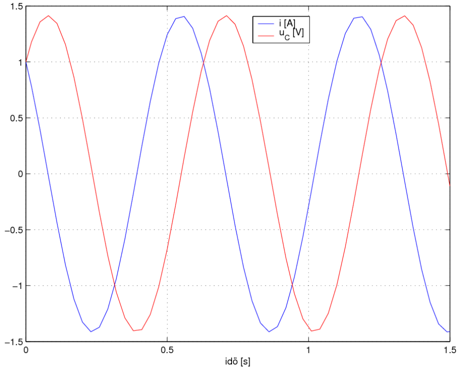
Example: asymptotic stability

RLC circuit, parameters: $R = 1\ \Omega$, $L = 10^{-1}H$, $C = 10^{-1}F$.
$u_C(0) = 1$ V, $i(0) = 1$ A, $u_{be}(t) = 0$ V



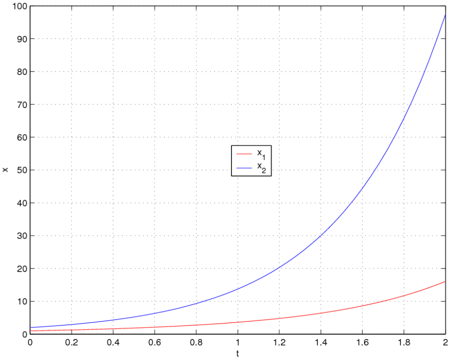
Non-asymptotic stability

(R)LC circuit, parameters: $R = 0\ \Omega(!)$, $L = 10^{-1} H$, $C = 10^{-1} F$.
$u_C(0) = 1$ V, $i(0) = 1$ A, $u_{be}(t) = 0$ V



Example: instability

$$\begin{aligned} \dot{x}_1 &= x_1 + 0.1x_2 \\ \dot{x}_2 &= -0.2x_1 + 2x_2 \end{aligned}, \quad x(0) = [12]^T$$



Overview

Stability of CT-LTI systems

- (Truncated) LTI state equation with ($u \equiv 0$):

$$\dot{x} = A \cdot x, \quad x \in \mathbb{R}^n, \; A \in \mathbb{R}^{n \times n}, \; x(0) = x_0$$

- Equilibrium pont: $x^* = 0$
- Solution:

$$x(t) = e^{At} \cdot x_0$$

- Recall: A diagonalizable (there exists invertible T, such that

$$T \cdot A \cdot T^{-1}$$

is diagonal) if and only if, A has n linearly independent eigenvectors.

Asymptotic stability of LTI systems – 1

Stability types:

- the real part of every eigenvalue of A is negative (A is a *stability matrix*): **asymptotic stability**
- A has eigenvalues with zero and negative real parts
  - the eigenvectors related to the zero real part eigenvalues are linearly independent: **(non-asymptotic) stability**
  - the eigenvectors related to the zero real part eigenvalues are not linearly independent: **(polynomial) instability**
- A has (at least) an eigenvalue with positive real part: **(exponential) instability**

Theorem

The eigenvalues of a square $A \in \mathcal{R}^{nxn}$ matrix remain unchanged after a similarity transformation on A by a transformation matrix T:

$$A' = TAT^{-1}$$

Proof:

Let us start with the eigenvalue equation for matrix A

$$A\xi = \lambda\xi \ , \ \xi \in \mathcal{R}^n \ , \ \lambda \in \mathbb{C}$$

If we transform it using $\xi' = T\xi$ then we obtain

$$TAT^{-1}T\xi = \lambda T\xi$$

$$A'\xi' = \lambda\xi'$$

Asymptotic stability of LTI systems – 3

Theorem

A CT-LTI system is asymptotically stable iff A is a stability matrix.

Sketch of *Proof*: Assume A is diagonalizable

$$\bar{A} = TAT^{-1} = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ & & \ddots & 0 \\ 0 & \dots & 0 & \lambda_n \end{bmatrix}$$

$$\bar{x}(t) = e^{\bar{A}t} \cdot \bar{x}_0 \quad , \quad e^{\bar{A}t} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \dots & 0 \\ 0 & e^{\lambda_2 t} & \dots & 0 \\ & & \ddots & 0 \\ 0 & \dots & 0 & e^{\lambda_n t} \end{bmatrix}$$

BIBO and asymptotic stability

Theorem

Asymptotic stability implies BIBO stability for LTI systems.

Proof:

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau, \quad y(t) = Cx(t)$$

$$\begin{aligned} ||x(t)|| &\leq ||e^{At}x(t_0) + M \textstyle\int_0^t e^{A(t-\tau)}Bd\tau|| = \\ &= ||e^{At}(x(t_0) + M \textstyle\int_0^t e^{-A\tau}Bd\tau)|| = \\ &= ||e^{At}(x(t_0) + M[-A^{-1}e^{-A\tau}B]_0^t)|| = \\ &= ||e^{At}[x(t_0) - MA^{-1}e^{-At}B + MA^{-1}B]|| \end{aligned}$$

$$||x(t)|| \leq ||e^{At}(x(t_0) + MA^{-1}B) - MA^{-1}B||$$

Overview

Lyapunov theorem of stability

- Lyapunov-function: $V : \mathcal{X} \to \mathbb{R}$
  - $V > 0$, if $x \neq x^*$, $V(x^*) = 0$
  - V continuously differentiable
  - V non-increasing, i.e. $\frac{d}{dt}V(x) = \frac{\partial V}{\partial x}\dot{x} = \frac{\partial V}{\partial x}f(x) \leq 0$

Theorem (Lyapunov stability theorem)

- *If there exists a Lyapunov function to the system $\dot{x} = f(x)$, $f(x^*) = 0$, then x^* is a stable equilibrium point.*
- *If $\frac{d}{dt}V < 0$ then x^* is an asymptotically stable equilibrium point.*
- *If the properties of a Lyapunov function hold only in a neighborhood U of x^*, then x^* is a locally (asymptotically) stable equilibrium point.*

Lyapunov theorem – example

- System:

$$\dot{x} = -(x-1)^3$$

- Equilibrium point: $x^* = 1$
- Lyapunov function: $V(x) = (x-1)^2$

$$\frac{d}{dt}V = \frac{\partial V}{\partial x}\dot{x} = 2(x-1)\cdot(-(x-1)^3) =$$
$$= -2(x-1)^4 < 0$$

- The system is **globally asymptotically stable**

CT-LTI Lyapunov theorem – 1

Basic notions:

- $Q \in \mathbb{R}^{n \times n}$ **symmetric matrix**: $Q = Q^T$, i.e. $[Q]_{ij} = [Q]_{ji}$ (every eigenvalue of Q is real)
- symmetric matrix Q is **positive definite** ($Q > 0$): $x^T Q x > 0, \forall x \in \mathbb{R}^n$, $x \neq 0$ ($\Leftrightarrow$ every eigenvalue of Q is positive)
- symmetric matrix Q is **negative definite** $Q < 0$: $x^T Q x < 0, \forall x \in \mathbb{R}^n$, $x \neq 0$ ($\Leftrightarrow$ every eigenvalue of Q is negative)

Theorem (Lyapunov criterion for LTI systems)

The state matrix (A) of an LTI system is a stability matrix if and only if there exists a positive definite symmetric matrix P for every given positive definite symmetric matrix Q such that

$$A^T P + PA = -Q$$

CT-LTI Lyapunov theorem – 2

Proof:
$\Leftarrow$ Assume $\forall\ Q > 0\ \exists\ P > 0$ such that $A^T P + PA = -Q$. Let $V(x) = x^T Px$.

$$\frac{d}{dt}V = \dot{x}^T Px + x^T P\dot{x} = x^T(A^T P + PA)x < 0$$

$\Rightarrow$ Assume A is a stability matrix. Then

$$P = \int_0^\infty e^{A^T t} Q e^{At} dt$$

$$A^T P + PA = \int_\mathbf{0}^\infty A^T e^{A^T t} Q e^{At} dt + \int_\mathbf{0}^\infty e^{A^T t} Q e^{At} A dt = [e^{A^T t} Q e^{At}]_\mathbf{0}^\infty = \mathbf{0} - Q = -Q$$

Example: stability of RLC circuit – 1

Model ($x_1 = i_L$, $x_2 = u_C$, $u_{be} = 0$, $R = 1$, $C = 0.1$, $L = 0.05$):

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

eigenvalues of A (roots of $\equiv b(s)$): $-10 \pm 10i$
$\Rightarrow$ the RLC circuit is asymptotically stable

Example: stability of RLC circuit – 2

Lyapunov function: sum of kinetic and potential energies

$$V(x) = \frac{1}{2}(Lx_1^2 + Cx_2^2) = \frac{1}{2}x^T \begin{bmatrix} L & 0 \\ 0 & C \end{bmatrix} x$$

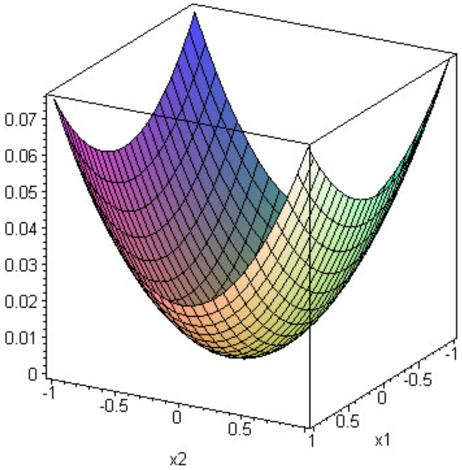
$$\frac{d}{dt}V = \frac{\partial V}{\partial x}\dot{x} = \frac{1}{2}(\dot{x}^T Px + x^T P\dot{x}) = -Rx_1^2$$

the sum of energies is not increasing (decreasing if $x_1 \neq 0$ and $R > 0$) independently of the actual values of the parameters

! the electric energy is preserved (is constant: $\frac{d}{dt}V = 0$), if $R = 0$.

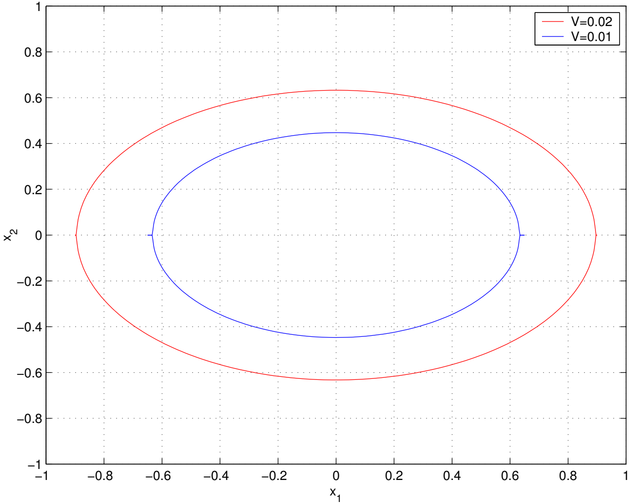
Example: stability of RLC circuit – 3

Plot of the Lyapunov function:



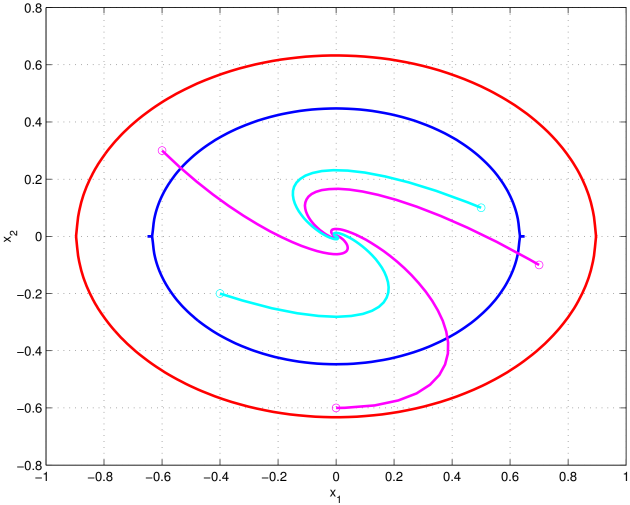
Example: stability of RLC circuit – 4

Level sets of the Lyapunov function (ellipses):



Example: stability of RLC circuit – 5

The solution of the ODE (voltages and currents) in the phase space:



Overview

Quadratic stability region

- Use quadratic Lyapunov function candidate with a given positive definite diagonal weighting matrix Q (tuning parameter!)

$$V\left[x(t)\right] = (x - x^*)^T \cdot Q \cdot (x - x^*)$$

- Dissipativity condition gives a conservative estimate of the stability region

$$\frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{dx}{dt} = \frac{\partial V}{\partial x}\overline{f}(x)$$

  - $\overline{f}(x) = f(x)$ in the open loop case with $u = 0$
  - $\overline{f}(x) = f(x) + g(x) \cdot C(x)$ in the closed-loop case where $C(x)$ is the static state feedback

Quadratic stability region: an example - 1

- Nonlinear system

$$\begin{aligned}\dot{x}_1 &= 0.4x_1x_2 - 1.5x_1\\ \dot{x}_2 &= -0.8x_1x_2 - 1.5x_2 + 1.5u\\ y &= x_2\end{aligned}$$

- Equilibrium point with $u^* = 7.75$

$$x^* = \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} 2 \\ 3.75 \end{bmatrix}$$

- Locally linearized system

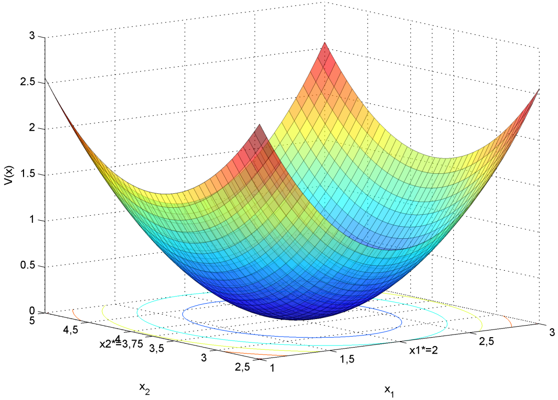
$$\begin{aligned}\dot{\tilde{x}} &= \begin{bmatrix} 0 & 0.8 \\ -3 & -3.1 \end{bmatrix}\tilde{x} + \begin{bmatrix} 0 \\ 1.5 \end{bmatrix}\tilde{u}\\ \tilde{y} &= \begin{bmatrix} 0 & 1 \end{bmatrix}\tilde{x}\end{aligned}$$

- Eigenvalues of the state matrix are $\lambda_1 = -1.5$ and $\lambda_2 = -1.6$ so equilibrium x^* (and not the whole system!) is locally asymptotically stable.

Quadratic stability region: an example - 2

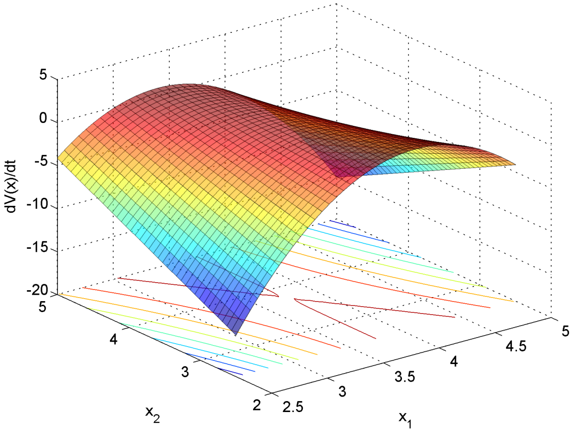
- Quadratic Lyapunov function

$$V(x) = (x - x^*)^T \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot (x - x^*)$$



Quadratic stability region: an example - 3

- Time derivative of the quadratic Lyapunov function



Summary

- stability analysis: how the system behaves if the stationary state is perturbed
- BIBO stability: only I/O property (boundedness of observed signals)
- stability in the state space: generally the property of the equilibrium point (there might be several with different stability properties)
- Lyapunov function: stability might be proved without knowing the solution
- stability of LTI state space models: system property
- checking stability of LTI systems: compute eigenvalues of $A \equiv$ poles of the transfer function