Computer controlled systems

Exercises for lecture 2 (Matlab practice: TP)

version: 2018.10.02. – 11:01:14

1 Basic linear algebra operations in Matlab (See also Appendix A)

1. Given an orthonormal basis of the image space of matrix
$$A = \begin{pmatrix} -2 & -1 & 7 \\ -3 & -5 & 14 \\ 1 & 2 & -5 \end{pmatrix}.$$
2. Give on orthonormal basis of the kernel (or null) space of matrix A.
3. Give on orthonormal basis of $\mathrm{Im}(A)^{\perp}$, the orthogonal complement of the image space of matrix A.
4. Give on orthonormal basis of the kernel space of matrix A^T.
5. Given on orthonormal basis of the image space of matrix A^T.
6. Compute the singular value decomposition of matrix A.
7. Compute the rank of diagonal matrix S and matrix A.
8. Compute the intersection of $V = \mathrm{Im}(A)$ and $W = \mathrm{Im}(A^T)$. Note that $V \cap W = \left(V^{\perp} \cup W^{\perp}\right)^{\perp}$
9. Give a matrix for which $\mathrm{Im}(A) = \mathrm{Ker}(A)$.
10. Compute the partial fraction decomposition of $H_1(s) = \frac{s}{s^2+3s+2}$ with `residue` and `partfrac`.
11. Compute the partial fraction decomposition of $H_2(s) = \frac{s^3+3s+1}{(s^2+4)(s-2)(s+3)(s^2+2s+2)}$ with ...

2 Dynamic system simulation in Matlab/Simulink

2.1 RLC circuit

The dynamics of the RLC circuit can be given by the following linear model:
$$\begin{cases} \dot{x}_1 = -\dfrac{R}{L}x_1 - \dfrac{1}{L}x_2 + \dfrac{1}{L}u & \text{(state equations)} \\ \dot{x}_2 = \dfrac{1}{C}x_1 & \\ y = x_2 & \text{(output equation)} \end{cases} \tag{1}$$
where $x_1 = i$, $x_2 = u_C$, $u = u_{\mathrm{in}}$ is the input, $y = x_2 = u_C$ is the output.

1. Simulate the system using `ode45`, with no input.
2. Compute the $H(s)$ transfer function of the system.
3. Compute impulse and step response of the system.
4. Simulate the system in Simulink with a given input.

2.2 Lotka Volterra model (simple ecological system)

In this section, I give two different predator-pray dynamics:

$$\Sigma_1 : \begin{cases} \dot{x}_1 = x_1(-x_1 + 2) \\ \dot{x}_2 = x_2(0.2x_1 - 1) \end{cases} \qquad \Sigma_2 : \begin{cases} \dot{x}_1 = x_1(-2x_1 - 3x_2 + 5) \\ \dot{x}_2 = x_2(1.4x_1 + x_2 - 2.4) \end{cases} \tag{2}$$

1. Compute the equilibrium points of this nonlinear system. You can use `fsolve`, which is function of the Symbolic Math Toolbox (SMT).
2. Determine which equilibrium points are stable.
3. Simulate the system with different initial conditions around the stable equilibrium point.

2.3 Continuous fermentation process

The dynamic equation of the continuous fermentation (or bioreactor) process is the following:

$$\begin{cases} \dot{x}_1 = \mu(x_2)x_1 - \dfrac{x_1 F}{V} \\ \dot{x}_2 = -\dfrac{\mu(x_2)x_1}{Y} + \dfrac{(S_F - x_2)F}{V} \end{cases} \quad \text{where} \quad \mu(x_2) = \frac{x_2 \mu_{\max}}{K_2 x_2^2 + x_2 + K_1} \tag{3}$$

The value of the constants F, V, Y, S_F, K_1, K_2, $\mu_{\max}$ are given in the Matlab script.

1. Compute the equilibrium points of this nonlinear system. You can use `fsolve`, which is function of the Symbolic Math Toolbox (SMT).
2. Determine which equilibrium points are stable.
3. Simulate the system with different initial conditions around the stable equilibrium point.
4. Define a proper terminal condition for `ode45`:
   - Terminate if the solution is far enough from the equilibrium point.
   - Terminate if any of the state variables reaches zero.
5. Color the convergent and divergent trajectories with two different colors (eg. red if convergent, blue if divergent).
6. Simulate the dynamics in Simulink.

A Fundamental theorem of linear algebra

Theorem 1. The fundamental theorem of linear algebra

Let $\mathcal{A} : \mathbb{R}^n \to \mathbb{R}^m$, $\mathcal{A}(x) = Ax$, where $A \in \mathbb{R}^{m\times n}$. Then the followings are true

$$\operatorname{Im}(A) = \operatorname{Ker}\left(A^T\right)^{\perp} \subset \mathbb{R}^m \tag{4a}$$

$$\operatorname{Im}\left(A^T\right) = \operatorname{Ker}(A)^{\perp} \subset \mathbb{R}^n \tag{4b}$$

Furthermore

$$\operatorname{Im}(A) \otimes \operatorname{Ker}\left(A^T\right) = \mathbb{R}^m \tag{5a}$$

$$\operatorname{Im}\left(A^T\right) \otimes \operatorname{Ker}(A) = \mathbb{R}^n \tag{5b}$$

Remark. If $r = \operatorname{rank}(A)$, than

$$\dim \operatorname{Im}(A) = r, \qquad \dim \operatorname{Ker}\left(A^T\right) = m - r \tag{6a}$$

$$\dim \operatorname{Im}\left(A^T\right) = r, \qquad \dim \operatorname{Ker}(A) = n - r \tag{6b}$$

Proof. Proof of (4a) as presented in [1]. Let

$$A = \begin{pmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \dots & \boldsymbol{a}_n \end{pmatrix} \;\Rightarrow\; A^T = \begin{pmatrix} \boldsymbol{a}_1^T \\ \boldsymbol{a}_2^T \\ \vdots \\ \boldsymbol{a}_n^T \end{pmatrix} \tag{7a}$$

$$\boldsymbol{x} \in \operatorname{Ker}\left(A^T\right) \;\Rightarrow\; A^T\boldsymbol{x} = \begin{pmatrix} \boldsymbol{a}_1^T\boldsymbol{x} \\ \boldsymbol{a}_2^T\boldsymbol{x} \\ \vdots \\ \boldsymbol{a}_n^T\boldsymbol{x} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{7b}$$

$$\boldsymbol{y} \in \operatorname{Im}(A) \;\Rightarrow\; \exists \alpha_i \in \mathbb{R} \text{ such that } \boldsymbol{y} = \sum_{i=1}^{n} \alpha_i \boldsymbol{a}_i \tag{7c}$$

Note that $\boldsymbol{x}$ and $\boldsymbol{y}$ are arbitrary vector elements of $\operatorname{Ker}\left(A^T\right)$ and $\operatorname{Im}(A)$, respectively. Then we compute the dot product of $\boldsymbol{x}$ and $\boldsymbol{y}$:

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{y}^T\boldsymbol{x} = \sum_{i=1}^{n} \alpha_i \boldsymbol{a}_i^T \boldsymbol{x} = 0, \tag{8}$$

since $\boldsymbol{a}_i^T\boldsymbol{x} = 0$, $\forall i = \overline{1,n}$. Consequently, $\boldsymbol{x} \perp \boldsymbol{y}$ for all possible $x \in \operatorname{Ker}\left(A^T\right)$ and $y \in \operatorname{Im}(A)$, which means that the two subspaces are the ***orthogonal complement*** for each other:

$$\begin{aligned} \operatorname{Im}(A) &= \operatorname{Ker}\left(A^T\right)^{\perp} \\ \operatorname{Im}(A) \cap \operatorname{Ker}\left(A^T\right) &= \{0\} \end{aligned} \tag{9}$$

Following this idea, we can conclude that the subspaces are linearly independent, therefore,

$$\dim\left(\operatorname{Im}(A) \otimes \operatorname{Ker}\left(A^T\right)\right) = r + (m - r) = m. \tag{10}$$

This can only happend if ***direct product*** of the two spaces is $\mathbb{R}^m$, which completes the proof for (5a). □

Proposition 2. (Self-adjoint operator) Let $\mathcal{A} : \mathbb{R}^n \to \mathbb{R}^n$, $\mathcal{A}(x) = Ax$, where $A \in \mathbb{R}^{n\times n}$ is a symmetric matrix: $A = A^T$. Than, as a consequence of Theorem 1, we have that

$$\operatorname{Im}(A) = \operatorname{Ker}(A)^{\perp} \text{ and } \operatorname{Im}(A) \otimes \operatorname{Ker}(A) = \mathbb{R}^n.$$

For more, see [2, Eq. (10.3)].

Proposition 3. Singular value decomposition (SVD)

If we make the SVD for matrix $A \in \mathbb{R}^{m\times n}$

$$A = U\Sigma V^T, \tag{11}$$

where

$$U \in \mathbb{R}^{m\times m} \text{ is unitary: } U^*U = I_m \tag{12a}$$
$$V \in \mathbb{R}^{n\times n} \text{ is unitary: } V^*V = I_n \tag{12b}$$
$$\Sigma \in \mathbb{R}^{m\times n} \text{ eigenvalues in the diagonal.} \tag{12c}$$

After this decomposition, the basis of the four subspaces (5) can be obtained as presented below.

$$\begin{aligned}
\mathrm{Im}(A):&\quad \text{the first } r \text{ colums of } U\\
\mathrm{Ker}(A^T):&\quad \text{the last } m-r \text{ columns of } U\\
\mathrm{Im}(A^T):&\quad \text{the first } r \text{ columns of } V\\
\mathrm{Ker}(A):&\quad \text{the last } n-r \text{ columns of } V
\end{aligned}$$

In short

$$\text{“ } A = \left[\mathrm{Im}(A) \quad \mathrm{Ker}(A^T)\right] \Sigma \left[\mathrm{Im}(A^T) \quad \mathrm{Ker}(A)\right]^T \text{ ”} \tag{13}$$

References

[1] Alexey Grigorev. The Fundamental Theorem of Linear Algebra. Technische Universität Berlin.

[2] Lantos Béla. *Irányítási rendszerek elmélete és tervezése I.* Akadémiai Kiadó Budapest, 2001.