Computer Controlled Systems

Supplementary homework for TP students

(i.e. compulsory only for TP students)

Submission deadline: **11th of December**, at 17:15

All solutions are expected to be calculated in Matlab. To model and solve linear matrix inequalities, we advise to use YALMIP, with SeDuMi solver. In order to ease your computations, any built-in Matlab functions are allowed to be used, including Symbolic Math Toolbox.

Problems

It is given the following linear parameter varying (LPV) system:

$$\mathcal{S}: \begin{cases} \dot{x} = A(\varrho)\,x + B(\varrho)\,u & x \in \mathbb{R}^5,\ u \in \mathbb{R} \\ y = C\,x & y \in \mathbb{R} \end{cases} \tag{1}$$

where $\varrho \in \mathbb{R}^2$ denotes the uncertain parameters:

$$\begin{aligned} A(\varrho) &= A_0 + A_1\varrho_1 + A_2\varrho_2, \quad \varrho_1 \in [\underline{\varrho}_1, \overline{\varrho}_1], \\ B(\varrho) &= B_0 + B_1\varrho_1 + B_2\varrho_2, \quad \varrho_2 \in [\underline{\varrho}_2, \overline{\varrho}_2], \end{aligned} \tag{2}$$

and matrices A_0, A_1, A_2, B_0, B_1 and B_2 are the following:

$$A_0 = \begin{pmatrix} -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & -5 & 0 & 52 \\ 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 21 \end{pmatrix},\ A_1 = \begin{pmatrix} 0.5 & 0 & 0 & -0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 2 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},\ A_2 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$B_0 = \begin{pmatrix} -6 \\ 3 \\ 3 \\ -6 \\ 1 \end{pmatrix},\ B_1 = \begin{pmatrix} -2 \\ 1 \\ 1 \\ -2 \\ 0 \end{pmatrix},\ B_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix},\ C = \begin{pmatrix} 4 & 0 & -2 & -4 & 7 \end{pmatrix},$$

Part I. Let $\varrho_1 = 0$ and $\varrho_2 = 0.5$, i.e. system $\mathcal{S}$ reduces to a simple LTI system.

1. Producing the controllability/observability staircase form with Matlab, check whether the system is controllable/stabilisable and observable/detectable.

   Hint. We recommend to consider the singular value decomposition (SVD) of controllability matrix $\mathcal{C}_n$ and the observability matrix $\mathcal{O}_n$.

2. Using linear matrix inequalities, try to find a static state feedback gain K, such that $u = -Kx$, and the closed loop dynamics $\dot{x} = (A - BK)x$ is asymptotically stable by the means of a quadratic Lyapunov function $V(x) = x^T P x$.

   Hint. You need to consider a simplified version of the technique present in the *theoretical background* of Part II.1. Since the value of ϱ is given exactly, you need to solve only:

$$\begin{cases} QA^T(\varrho) + A(\varrho)Q - N^T B^T(\varrho) - B(\varrho)N \prec 0, \text{ where } \varrho = (0, 0.5) \\ \text{But do not forget: } Q \succ 0 \end{cases} \tag{3}$$

3. Using linear matrix inequalities, try to find an observer gain L, such that the observer's dynamics is $\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y})$, and the error dynamics $\dot{e} = \dot{x} - \dot{\hat{x}} = (A - LC)e$ is asymptotically stable by the means of a quadratic Lyapunov function $V(x) = x^T P x$.

   Hint. Try to introduce an auxiliary matrix variable: $M = PL$

Part II. Let $\varrho_1 \in [-1, 1]$ and $\varrho_2 \in [-0.5, 0.5]$.

1. Using linear matrix inequalities, try to find a static state feedback gain K, such that $u = -Kx$, and the closed loop dynamics $\dot{x} = (A(\varrho) - B(\varrho)K)x$ is asymptotically stable by the means of a quadratic Lyapunov function $V(x) = x^T P x$ for any admissible parameter value.

   Theoretical background. The closed loop dynamics is asymptotically stable in the Lyapunov sense if function $V(x)$ is a proper Lyapunov function, namely:

   (a) $V(x) > 0 \quad \Leftrightarrow \quad P = P^T \succ 0$, i.e. P is a positive definite symmetric matrix

   (b) $\dot{V}(x) = \dot{x}^T P x + x^T P \dot{x} = x^T \Big((A(\varrho) - B(\varrho)K)^T P + P(A(\varrho) - B(\varrho)K) \Big) x < 0$, equivalently:

   $$(A(\varrho) - B(\varrho)K)^T P + P(A(\varrho) - B(\varrho)K) \prec 0, \quad \forall \varrho \in \mathcal{R}, \tag{4}$$

   where $\forall \varrho \in \mathcal{R}$ means that $\forall (\varrho_1, \varrho_2) \in [\underline{\varrho}_1, \overline{\varrho}_1] \times [\underline{\varrho}_2, \overline{\varrho}_2] = [-1, 1] \times [-0.5, 0.5]$.

   Note that both P and K are free (unknown) matrix variables of the optimization problem, therefore, the previous matrix inequality (4) is ***bilinear*** in P and K. In order to make it linear, we multiply it from both sides by $Q = P^{-1} \succ 0$, then we obtain:

   $$QA^T(\varrho) + A(\varrho)Q - QK^T B^T(\varrho) - B(\varrho)KQ \prec 0, \quad \forall \varrho \in \mathcal{R} \tag{5}$$

   Let us introduce the auxiliary matrix variable $N = KQ$, then the final inequality

   $$QA^T(\varrho) + A(\varrho)Q - N^T B^T(\varrho) - B(\varrho)N \prec 0, \quad \forall \varrho \in \mathcal{R} \tag{6}$$

   is a parameter ϱ dependent linear matrix inequality (LMI) in the unknown matrix variables $Q = Q^T \succ 0$ and N.

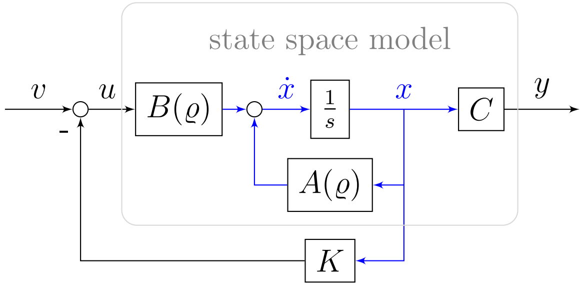
   In order to check the feasibility of (6), it is enough to solve a system of parameter *independent* LMIs in the corner points of domain $\mathcal{R}$, namely:

   $$\begin{cases} QA^T(\underline{\varrho}_1, \underline{\varrho}_2) + A(\underline{\varrho}_1, \underline{\varrho}_2)Q - N^T B^T(\underline{\varrho}_1, \underline{\varrho}_2) - B(\underline{\varrho}_1, \underline{\varrho}_2)N \prec 0 \\ QA^T(\overline{\varrho}_1, \underline{\varrho}_2) + A(\overline{\varrho}_1, \underline{\varrho}_2)Q - N^T B^T(\overline{\varrho}_1, \underline{\varrho}_2) - B(\overline{\varrho}_1, \underline{\varrho}_2)N \prec 0 \\ \dots \\ \text{Do not forget: } Q \succ 0 \end{cases} \tag{7}$$

   After that Q and N are determined, matrix P and K can be simply computed.

2. Compute the induced $\mathcal{L}_2$ operator gain for both systems $\mathcal{S} : (A(\varrho), B(\varrho), C)$ and $\mathcal{S}_{\text{feedback}} : (A(\varrho) - B(\varrho)K, B(\varrho), C)$, where gain K was computed in the previous point.

3. Consider the block diagram below, and compute the optimal feedback gain K, $u = -Kx + v$ (where v is the disturbance input), that stabilizes the system and gives a minimal $\mathcal{L}_2$ gain from the disturbance v to the output y.


state space model
v
u
-
B(ϱ)
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1/s
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