Computer Controlled Systems

Homework 4.

Submission deadline: **13th of December**, at 13:00 (approx. 2 weeks)

All solutions are expected to be calculated by hand, also all figures have to be drawn by hand. Computer programs (e.g. Matlab) can be used for self-verification, but all problems have to contain the detailed steps of solutions

Problems

1. Design an optimal LQR controller $u(t)$ for the following system:

$$\dot{x} = Ax + Bu, \text{ where } A = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} \\ 0 & -\frac{1}{4} \end{pmatrix}, B = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}.$$

that minimizes the cost function

$$J(x,u) = \frac{1}{2}\int_0^\infty x^T(t)Qx(t) + u^T(t)Ru(t)\,\mathrm{d}t. \tag{1}$$

   (a) The continuous-time algebraic Riccati equation (CARE) may have multiple solutions. Which of them will result in a stable closed loop dynamics?

   (b) In order to check your solution, you can use function `care` of Matlab's Control System Toolbox.

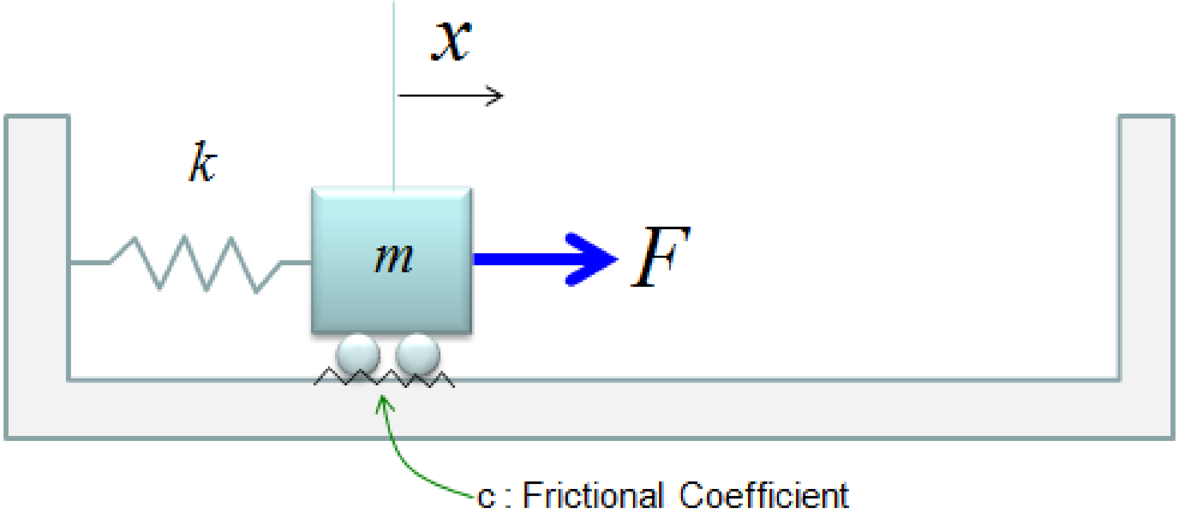

Figure 1: Spring-mass system with friction.

2. Consider a simple mass-spring-dumper system, illustrated by the picture above. We can describe the system with one differential equation by using Newton's second law $\sum_{i=1}^{n} F_i = m\frac{\mathrm{d}v}{\mathrm{d}t}$. Moreover we know that the velocity is the first order derivative of the position. Applying these, we get the following system

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = v \\ m\dfrac{\mathrm{d}v}{\mathrm{d}t} = -kx - cv + F \end{cases} \tag{2}$$

where k is the spring constant and c is the frictional coefficient. Let us choose x and v as state variables, and rename them as x_1 and x_2. The external force F is the input of the system. Now the equations are the followings

$$\begin{cases} \dot{x_1} = x_2 \\ \dot{x_2} = \dfrac{-kx_1}{m} - \dfrac{-cx_2}{m} + \dfrac{F}{m} \end{cases} \tag{3}$$

We would like to measure the position of the system $y = x$. Now we can rewrite the system in matrix-vector form

$$\begin{cases} \dot{x} = \begin{pmatrix} 0 & 1 \\ \frac{-k}{m} & \frac{-c}{m} \end{pmatrix} x + \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix} F \\ y = \begin{pmatrix} 1 & 0 \end{pmatrix} x \end{cases} \tag{4}$$

(a) Rewrite the system in matrix-vector form with the following parameters $m = 0.1$, $k = 0.4$, $c = 0$.

(b) Compute the model matrices (Φ and Γ) of the discrete-time (DT) state space model of the mass-spring-dumper if the sampling time is $h = \frac{\pi}{12}$!

(c) Compute the eigenvalues of DT state transition matrix (Φ) and determine whether the system is stable or not?

(d) Determine the value of the state vector of the DT system in the following sampling points: $x(2)$, if the input is $u(k) = 2$ for all $k = 0, 1, ...$ and the initial state is $x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

3. Given the following DT-LTI system

$$\begin{cases} x(k+1) = \begin{pmatrix} 2 & 0 \\ 4 & 3 \end{pmatrix} x(k) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(k) \\ y(k) = \begin{pmatrix} 1 & -1 \end{pmatrix} x(k) \end{cases} \tag{5}$$

(a) Give the transfer operator $H(q)$ of the system!

(b) Is the system controllable and reachable?

(c) Is the system observable?