1 Homework

1.1 problem

$$\dot{x}_1 = -\frac{1}{C}x_2 + \frac{1}{C}u$$
$$\dot{x}_2 = \frac{1}{L}x_1 - \frac{R}{L}x_2$$
$$y = Rx_2$$

1.1.1 (a)

With

$$A = \begin{bmatrix} 0 & -\frac{1}{C} \\ \frac{1}{L} & -\frac{R}{L} \end{bmatrix} \qquad B = \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix}$$
$$C = \begin{bmatrix} 0 & R \end{bmatrix} \qquad D = 0$$

and $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du.$$

1.1.2 (b)

$$H(s) = C(sI - A)^{-1}B + D = \frac{R}{CLs^2 + CRs + 1}$$

1.1.3 (c)

$$h(t) = \mathcal{L}^{-1}\{H(s)\} = 100e^{-2t}\sin 4t$$

1.1.4 (d)

Since $A \in \mathbb{R}^{2\times 2}$ the eigenvalues can be calculated from the equation

$$\lambda^2 - \operatorname{tr} A\lambda + \det A = 0 \implies \lambda_{1,2} = -2 \pm 4j$$

1.1.5 (e)

For

$$V(x) = E_L + E_C = \frac{1}{2}Lx_1^2 + \frac{1}{2}Cx_2^2$$

we have $V(0) = 0$, $V(x) > 0$ for all $x \neq 0 \in \mathbb{R}^2$ since $L, C > 0 \in \mathbb{R}$. Furthermore we have

$$\dot{V}(x) = \left\langle \frac{\partial V}{\partial x}, \dot{x} \right\rangle = \begin{bmatrix} Cx_1 & Lx_2 \end{bmatrix} Ax = -Rx_2^2$$

thus $\dot{V}(x) < 0$ for all $x \neq 0 \in \mathbb{R}^2$. This means that V is an appropriate Lyapunov function.

1.2 problem

$$\dot{x}_1 = -k_1x_1$$
$$\dot{x}_2 = k_1x_1 - k_2x_2 + u$$
$$\dot{x}_3 = k_2x_2 - k_3x_3$$
$$\dot{x}_4 = k_3x_3 - k_4x_4$$
$$y = x_3$$

1.2.1 (a)

With

$$A = \begin{bmatrix} -k_1 & 0 & 0 & 0 \\ k_1 & -k_2 & 0 & 0 \\ 0 & k_2 & -k_3 & 0 \\ 0 & 0 & k_3 & -k_4 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$
$$C = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \qquad D = 0$$

and $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du.$$

1.2.2 (b)

$$C_4 = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & -k_2 & k_2^2 & -k_2^3 \\ 0 & k_2 & -k_2^2 - k_2k_3 & k_2^3 + k_2^2k_3 + k_2k_3^2 \\ 0 & 0 & k_2k_3 & -k_2^2k_3 - k_2k_3^2 - k_2k_3k_4 \end{bmatrix}$$

$$O_4 = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & k_2 & -k_3 & 0 \\ k_1k_2 & -k_2^2 - k_2k_3 & k_3^2 & 0 \\ -k_1^2k_2 + k_1k_2^2 - k_1k_2k_3 & k_2^3 + k_2^2k_3 + k_2k_3^2 & -k_3^3 & 0 \end{bmatrix}$$

1.2.3 (c)

It is easy to see that $\operatorname{rank} C_4 = 3$, thus

$$X_c = \operatorname{Im} C_4 = \operatorname{span}\{B, AB, A^2B\}.$$

Similarly $\operatorname{rank} O_4 = 3$, thus

$$X_{\overline{o}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

1.2.4 (d)

1. The controllability subspace is the set of state vectors, which can be reached in finite time with some input, i.e. for any input the systemt stays in this subspace. This makes sense, since in the first compartment the amount of water cannot be influenced by the input.
2. The unobservability subspace is the set of initial values, which cannot be distinguished from each other measuring the output. This means that if only the fourth compartment had water in it, than the output would be the same. This makes sense, since from the fourth compartment the liquid just disappears from the system without incluencing the output.

1.2.5 (e)

We can easily see that

$$A_m = \begin{bmatrix} -k_2 & 0 \\ k_2 & -k_3 \end{bmatrix} \qquad B_m = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$C_m = \begin{bmatrix} 0 & 1 \end{bmatrix} \qquad D_m = 0$$

is a minimal realization, since it is jointly controllable and observable. To be sure we can check that

$$H(s) = C(sI - A)^{-1}B + D = C_m(sI - A_m)^{-1}B_m + D_m = \frac{k_2}{(s+k_2)(s+k_3)}.$$