1 Homework

1.1 problem

The given linear mapping is $\mathcal{A} : \mathbb{R}^3 \mapsto \mathbb{R}^3$, $\mathcal{A}(v) = Av$, where

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

(a)

We apply the mapping to the vector $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \in \mathbb{R}^3$.

$$\mathcal{A}(v) = Av = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_2 + v_3 \\ v_1 + v_3 \\ v_1 + v_2 \end{pmatrix}.$$

(b)

The characteristic equation and the eigenvalues are

$$\det(\lambda I - A) = \begin{vmatrix} \lambda & -1 & -1 \\ -1 & \lambda & -1 \\ -1 & -1 & \lambda \end{vmatrix} = \lambda^3 - 3\lambda - 2 = (\lambda - 2)(\lambda + 1)^2 = 0 \implies \lambda_1 = 2 \ \lambda_{2,3} = -1.$$

1.

$$(\lambda_1 I - A)x = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_1 - x_2 - x_3 \\ -x_1 + 2x_2 - x_3 \\ -x_1 - x_2 + 2x_3 \end{pmatrix} = 0$$

From this we get

$$\begin{aligned} 2x_1 - x_2 - x_3 &= 0 \\ \frac{3}{2}x_2 - \frac{3}{2}x_3 &= 0 \end{aligned}$$

and $x_1 = x_2 = x_3$, which means that the eigenvector corresponding to $\lambda_1 = 2$ is $v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and the eigenspace is

$$\operatorname{span}\{v_1\} = \left\{ pv_1 \middle| p \in \mathbb{R} \backslash \{0\} \right\}.$$

2.

$$(\lambda_{2,3} I - A)x = \begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

From this we get

$$x_1 = -x_2 - x_3.$$

This means that the eigenvectors corresponding to $\lambda_{2,3} = -1$ are $v_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ and $v_3 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ and the eigenspace is

$$\operatorname{span}\{v_1, v_2\} = \left\{ pv_2 + qv_3 \middle| p, q \in \mathbb{R},\ p + q \neq 0 \right\}.$$

(c)

The matrix

$$S = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

is an appropriate coordinate transformation matrix and

$$D = S^{-1}AS = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

(d)

Since D is a diagonal matrix

$$e^D = \begin{pmatrix} e^2 & 0 & 0 \\ 0 & \frac{1}{e} & 0 \\ 0 & 0 & \frac{1}{e} \end{pmatrix}.$$

(e)

$$e^A = Se^DS^{-1} = \begin{pmatrix} \frac{e^3+2}{3e} & \frac{e^3-1}{3e} & \frac{e^3-1}{3e} \\ \frac{e^3-1}{3e} & \frac{e^3+2}{3e} & \frac{e^3-1}{3e} \\ \frac{e^3-1}{3e} & \frac{e^3-1}{3e} & \frac{e^3+2}{3e} \end{pmatrix}$$

1.2 problem

The given matrix is

$$B = \begin{pmatrix} 1 & -1 & 0 & 22 \\ 0 & 1 & -2 & 5 \\ -3 & 2 & 5 & -65 \\ -2 & 6 & 4 & 0 \end{pmatrix}.$$

We need to solve the linear equation system

$$Bx = \begin{pmatrix} x_1 - x_2 + 22x_4 \\ x_2 - 2x_3 + 5x_4 \\ -3x_1 + 2x_2 + 5x_3 - 65x_4 \\ -2x_1 + 6x_2 + 4x_3 \end{pmatrix} = 0.$$

After Gaussian elimination we get

$$\begin{aligned} x_1 - x_2 + 22x_4 &= 0 \\ x_2 - 2x_3 + 5x_4 &= 0 \\ 3x_3 + 6x_4 &= 0. \end{aligned}$$

From this we get $x_3 = -2x_4$, $x_2 = -9x_4$ and $x_1 = -31x_4$ which means that any $v = \begin{pmatrix} -31x_4 \\ -9x_4 \\ -2x_4 \\ x_4 \end{pmatrix} = x_4 \begin{pmatrix} -31 \\ -9 \\ -2 \\ 1 \end{pmatrix}$ is a solution of the linear equation system $Bx = 0$, $\forall x_4 \in \mathbb{R}$. Using this

$$\begin{aligned} \mathrm{Im}(B) &= \mathrm{span}\,\{b_1, b_2, b_3\} \\ \mathrm{Ker}(B) &= \mathrm{span}\,\{v\} \end{aligned}$$

where $B = \begin{pmatrix} b_1 & b_2 & b_3 & b_4 \end{pmatrix}$.

1.3 problem

$$\ddot{y} + 4\dot{y} + 3y = u(t)$$

where $\dot{y}(0) = -1$ and $y(0) = 1$. We know that

$$\mathcal{L}\{\ddot{y}\} = s^2Y(s) - sy(0) - \dot{y}(0) = s^2Y(s) - s + 1$$
$$\mathcal{L}\{\dot{y}\} = sY(s) - y(0) = sY(s) - 1.$$

Using this the above equation can be written as

$$s^2Y(s) - s + 1 + 4sY(s) - 4 + 3Y(s) = \left(s^2 + 4s + 3\right)Y(s) - s - 3 = U(s) \implies Y(s) = \frac{U(s) + s + 3}{(s+1)(s+3)}.$$

(a)

With $u(t) = 4e^{-2t}$ we get $U(s) = \frac{4}{s+2}$ and

$$Y(s) = \frac{s^2 + 5s + 10}{(s+1)(s+2)(s+3)} \implies y(t) = 3e^{-t} - 4e^{-2t} + 2e^{-3t}.$$

(b)

With $u(t) = \sin 5t$ we get $U(s) = \frac{5}{s^2+25}$ and

$$Y(s) = \frac{s^3 + 3s^2 + 25s + 80}{(s+1)(s+3)(s^2+25)} \implies y(t) = \frac{57}{52}e^{-t} - -\frac{5}{68}e^{-3t} - \frac{5}{221}\cos 5t - \frac{11}{442}\sin 5t.$$

(c)

With $u(t) = e^{-2t}\cos 5t$ we get $U(s) = \frac{s+2}{(s+2)^2+25}$ and

$$Y(s) = \frac{s^3 + 7s^2 + 42s + 89}{(s+1)(s+3)(s^2+4s+29)} \implies y(t) = \frac{53}{52}e^{-t} + \frac{1}{52}e^{-3t} - \frac{1}{26}e^{-2t}\cos 5t.$$

1.4 problem

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

where

$$A = \begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad C = \begin{pmatrix} 1 - 1 \end{pmatrix}.$$

(a)

$$H(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B = \frac{s-1}{s^2 - 3s + 1}$$

(b)

$$h(t) = \mathcal{L}^{-1}\{H(s)\} = e^{\frac{3}{2}t}\left(\cosh\frac{\sqrt{5}}{2}t + \frac{1}{\sqrt{5}}\sinh\frac{\sqrt{5}}{2}t\right)$$