Computer controlled systems

Crane model

version: 2017.11.18. – 15:06:48

1 Crane model (rakodó darumodell)

Consider the following mathematical model of a crane machine. For the sake of simplicity, we restrict the motion of the carried weight to the (y, z) plane. In the model we identify the following time-dependent (state-) variables:

- $R = R(t)$ denotes the actual position of the car on the rail,
- $L = L(t)$ denotes the actual length of the wire,
- $\theta = \theta(t)$ denotes the actual angle of the wire with the vertical.

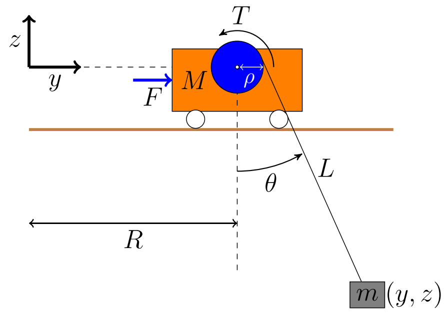

Figure 1. Kinetic model of the crane machine.

Know parameters (constants):

- ρ is the radius of the pulley[1]. In the geometrical expression the radius of the pulley is neglected, eg. if the angle is $\theta = 0$ and the position of the car is $R = 0$ than the position of the weight along the y axis is considered to be R.
- M is the mass of the car
- m is the mass of the lifted weight
- J is the moment of inertia[2] of the pulley.

Manipulate inputs are the following:

- F is the driving force applied to the car
- T is the torque applied to the pulley

Measured quantities: $R(t)$ and $L(t)$.

Tekintse az ábrán látható rakodó darut. Az egyszerűség kedvéért a teher mozgását az (y, z) függőleges síkra korlátozzuk. Jelölje R a sínen mozgó kocsi y irányú pozícióját, L a sodrony hosszát és θ a sodrony függőlegessel bezárt szögét. Ismert geometriai paraméter a sodronyt tekercselő dob sugara, melyet ρ-val jelöltünk. (A geometriai összefüggésekben a dob sugarát elhanyagoljuk, azaz $\theta = 0$ esetben $R = y$). Szintén ismertek az inercia paraméterek: M a kocsi tömege; m a mozgatott teher tömege és J a sodronyt tekercselő hajtás és a dob tehetetlenségi nyomatéka.

A beavatkozó jelek:

- F a kocsira ható erő
- T a sodronyt tekercselő dobra ható forgatónyomaték

A mért kimeneti változók: R és L.

The system's kinetic energy is a composition of the followings:

[1] sheave or drum: tekercselő csiga vagy tekercselő dob
[2] tehetetlenségi nyomaték

1. the kinetic energy of M is $T_M = \dfrac{M\dot{R}^2}{2}$
2. the kinetic energy of m is $T_m = \dfrac{mv^2}{2}$
3. the kinetic energy of the pulley is $T_J = \dfrac{J\dot{\theta}^2}{2} = \dfrac{J\dot{L}^2}{2\rho^2}$

The system's potential energy is the potential energy of m, that is $V_m = mgL(1-\cos\theta)$.

The system's Lagrangian is

$$\mathcal{L} = T - V = \frac{M\dot{R}^2}{2} + \frac{mv^2}{2} + \frac{J\dot{L}^2}{2\rho^2} - mgL(1-\cos\theta) \tag{1}$$

The velocity $\boldsymbol{v}$ of m has the following components (see Figure 1):

$$\boldsymbol{v} = L\dot{\theta}\boldsymbol{e}_t + \dot{R}\boldsymbol{e}_x + \dot{L}\boldsymbol{e}_n \tag{2}$$

Since $\boldsymbol{e}_t \perp \boldsymbol{e}_n$, the square of the norm of $\boldsymbol{v}$ can be computed in the following way

$$\begin{aligned} v = \|\boldsymbol{v}\|^2 = \langle \boldsymbol{v}, \boldsymbol{v}\rangle &= L^2\dot{\theta}^2 + \dot{R}^2 + \dot{L}^2 + 2L\dot{\theta}\dot{R}\langle \boldsymbol{e}_t, \boldsymbol{e}_x\rangle + 2\dot{R}\dot{L}\langle \boldsymbol{e}_n, \boldsymbol{e}_x\rangle \\ &= L^2\dot{\theta}^2 + \dot{R}^2 + \dot{L}^2 + 2L\dot{\theta}\dot{R}\cos\theta + 2\dot{R}\dot{L}\cos\left(\frac{\pi}{2}-\theta\right) \\ &= L^2\dot{\theta}^2 + \dot{R}^2 + \dot{L}^2 + 2L\dot{\theta}\dot{R}\cos\theta + 2\dot{R}\dot{L}\sin\theta \end{aligned} \tag{3}$$

Therefore, the Lagrangian can be written in the following form:

$$\mathcal{L} = \frac{(M+m)\dot{R}^2}{2} + \frac{mL^2\dot{\theta}^2}{2} + \frac{m\dot{L}^2}{2} + mL\dot{\theta}\dot{R}\cos\theta + m\dot{R}\dot{L}\sin\theta + \frac{J\dot{L}^2}{2\rho^2} - mgL(1-\cos\theta) \tag{4}$$

The Euler-Lagrange equations are the following:

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{R}} - \frac{\partial\mathcal{L}}{\partial R} = \frac{\mathrm{d}}{\mathrm{d}t}\Big((M+m)\dot{R} + mL\dot{\theta}\cos\theta + m\dot{L}\sin\theta\Big) \\ &= (M+m)\ddot{R} + m\dot{L}\dot{\theta}\cos\theta + mL\ddot{\theta}\cos\theta - mL\dot{\theta}^2\sin\theta + m\ddot{L}\sin\theta + m\dot{L}\dot{\theta}\cos\theta \\ &= (M+m)\ddot{R} + 2m\dot{L}\dot{\theta}\cos\theta + mL\ddot{\theta}\cos\theta - mL\dot{\theta}^2\sin\theta + m\ddot{L}\sin\theta \end{aligned} \tag{A1}$$

Second equation:

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{L}} - \frac{\partial\mathcal{L}}{\partial L} = \frac{\mathrm{d}}{\mathrm{d}t}\left(m\dot{L} + m\dot{R}\sin\theta + \frac{J\dot{L}}{\rho^2}\right) - mL\dot{\theta}^2 - m\dot{\theta}\dot{R}\cos\theta + mg(1-\cos\vartheta) \\ &= \left(m + \frac{J}{\rho^2}\right)\ddot{L} + m\ddot{R}\sin\theta + m\dot{R}\dot{\theta}\cos\theta - mL\dot{\theta}^2 - m\dot{\theta}\dot{R}\cos\theta + mg(1-\cos\vartheta) \\ &= \left(m + \frac{J}{\rho^2}\right)\ddot{L} + m\ddot{R}\sin\theta - mL\dot{\theta}^2 + mg(1-\cos\vartheta) \end{aligned} \tag{A2}$$

Third equation:

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} - \frac{\partial\mathcal{L}}{\partial\theta} = \frac{\mathrm{d}}{\mathrm{d}t}\Big(mL^2\dot{\theta} + mL\dot{R}\cos\theta\Big) + mL\dot{\theta}\dot{R}\sin\theta - m\dot{R}\dot{L}\cos\theta + mgL\sin\theta \\ &= 2mL\dot{L}\dot{\theta} + mL^2\ddot{\theta} + m\dot{L}\dot{R}\cos\theta + mL\ddot{R}\cos\theta - mL\dot{R}\dot{\theta}\sin\theta + mL\dot{\theta}\dot{R}\sin\theta - m\dot{R}\dot{L}\cos\theta + mgL\sin\theta \\ &= 2mL\dot{L}\dot{\theta} + mL^2\ddot{\theta} + mL\ddot{R}\cos\theta + mgL\sin\theta \end{aligned}$$

Dividing by L we get:

$$0 = 2m\dot{L}\dot{\theta} + mL\ddot{\theta} + m\ddot{R}\cos\theta + mg\sin\theta \tag{A3}$$

In equilibrium point the torque is $T_0 = mg\rho$. Let $T = T_0 + \tau$ be the net torque. Furthermore, we consider an F external force applying to M. Therefore, the equations of motion could be written as follows:

$$\begin{aligned} F &= (M+m)\ddot{R} + 2m\dot{L}\dot{\theta}\cos\theta + mL\ddot{\theta}\cos\theta - mL\dot{\theta}^2\sin\theta + m\ddot{L}\sin\theta \\ -\frac{T}{\rho} = -\frac{mg+\tau}{\rho} &= \left(m + \frac{J}{\rho^2}\right)\ddot{L} + m\ddot{R}\sin\theta - mL\dot{\theta}^2 - mg\cos\vartheta \\ 0 &= 2m\dot{L}\dot{\theta} + mL\ddot{\theta} + m\ddot{R}\cos\theta + mg\sin\theta \end{aligned} \tag{5}$$

We considering the following operating point parameter values (munkaponti paraméter értékek):

$$R_0, \quad L_0, \quad \Theta_0 = 0, \quad F_0 = 0, \quad T_0 = mg\rho \tag{6}$$

Than we center the input and state variables:

$$R = R_0 + r, \quad L = L_0 + l, \quad F = F_0 + f, \quad T = T_0 + \tau \tag{7}$$

Substituting these expressions into (5), we get:

$$\underbrace{f - 2m\dot{l}\dot{\theta}\cos\theta + m\dot{\theta}^2(L_0+l)\sin\theta}_{b_1} = \ddot{l}\,\underbrace{m\sin\theta}_{a_{11}} + \ddot{r}\,\underbrace{(M+m)}_{a_{12}} + \ddot{\theta}\,\underbrace{m(L_0+l)\cos\theta}_{a_{13}} \tag{8}$$

$$\underbrace{M(L_0+l)\dot{\theta}^2 - mg(1-\cos\theta) - \frac{\tau}{\rho}}_{b_2} = \ddot{l}\,\underbrace{\left(m + \frac{J}{\rho^2}\right)}_{a_{21}} + \ddot{r}\,\underbrace{m\sin\theta}_{a_{22}} \tag{9}$$

$$\underbrace{-2m\dot{l}\dot{\theta} - mg\sin\theta}_{b_3} = \ddot{r}\,\underbrace{m\cos\theta}_{a_{32}} + \ddot{\theta}\,\underbrace{m(L_0+l)}_{a_{33}} \tag{10}$$

Let us introduce the following state and input variables:

$$\begin{matrix} x_1 = l & x_3 = r & x_5 = \theta \\ x_2 = \dot{l} & x_4 = \dot{r} & x_6 = \dot{\theta} \end{matrix} \;\rightarrow\; x = \begin{pmatrix} x_1 \\ \dots \\ x_6 \end{pmatrix} \qquad \begin{matrix} u_1 = f \\ u_2 = \tau \end{matrix} \;\rightarrow\; u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{11}$$

Then, the resulting nonlinear state equation is the following

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_3 \\ \dot{x}_5 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_4 \\ x_6 \end{pmatrix} = \begin{pmatrix} f_1(x,u) \\ f_3(x,u) \\ f_5(x,u) \end{pmatrix}, \quad \begin{pmatrix} \dot{x}_2 \\ \dot{x}_4 \\ \dot{x}_6 \end{pmatrix} = \begin{pmatrix} \ddot{l} \\ \ddot{r} \\ \ddot{\theta} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & 0 \\ 0 & a_{32} & a_{33} \end{pmatrix}^{-1} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} f_2(x,u) \\ f_4(x,u) \\ f_6(x,u) \end{pmatrix} \tag{12}$$

Computing the matrix inverse, we get

$$\begin{pmatrix} f_2(x,u) \\ f_4(x,u) \\ f_6(x,u) \end{pmatrix} = \frac{1}{a_{11}a_{22}a_{33} - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32}} \begin{pmatrix} a_{13}a_{32}b_2 - a_{12}a_{33}b_2 - a_{13}a_{22}b_3 + a_{22}a_{33}b_1 \\ a_{13}a_{21}b_3 + a_{11}a_{33}b_2 - a_{21}a_{33}b_1 \\ a_{11}a_{22}b_3 - a_{12}a_{21}b_3 - a_{11}a_{32}b_2 + a_{21}a_{32}b_1 \end{pmatrix} \tag{13}$$

Therefore, the nonlinear state-space model can be written as follows:

$$\dot{x} = F(x,u), \quad \text{where} \quad F(x,u) = \begin{pmatrix} f_1(x,u) \\ f_2(x,u) \\ f_3(x,u) \\ f_4(x,u) \\ f_5(x,u) \\ f_6(x,u) \end{pmatrix} \tag{14}$$

In order to get a linearized model (in the operating point $(x_0 = 0, u_0 = 0)$, we considering the second order Taylor polynomial of $F(x,u)$:

$$F(x,u) \simeq F(x_0,u_0) + \left[\frac{\partial F(x,u)}{\partial x}\right]_{\substack{x=0 \\ u=0}} (x - x_0) + \left[\frac{\partial F(x,u)}{\partial u}\right]_{\substack{x=0 \\ u=0}} (u - u_0) \tag{15}$$

Since $F(x_0, u_0) = 0$, the linearized model is $\dot{x} = Ax + Bu$, where:

$$A = \left[\frac{\partial F(x,u)}{\partial x}\right]_{\substack{x=0\\u=0}} = \left(\begin{array}{cc|cccc} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{gm}{M} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -\frac{g(M+m)}{L_0 M} & 0 \end{array}\right) \quad B = \left[\frac{\partial F(x,u)}{\partial u}\right]_{\substack{x=0\\u=0}} = \left(\begin{array}{c|c} 0 & 0 \\ 0 & -\frac{\rho}{m\rho^2+J} \\ \hline 0 & 0 \\ \frac{1}{M} & 0 \\ 0 & 0 \\ -\frac{1}{L_0 M} & 0 \end{array}\right) \tag{16}$$

As one can immediately observe, we obtained two decoupled subsystems in the linearized model:

$$\begin{aligned} \dot{\xi}_1 &= A_1\xi_1 + B_1\tau, \quad \text{where} \quad A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad B_1 = \begin{pmatrix} 0 \\ -\frac{\rho}{m\rho^2+J} \end{pmatrix} \\ \dot{\xi}_1 &= A_2\xi_2 + B_2 f, \quad \text{where} \quad A_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{gm}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{g(M+m)}{L_0 M} & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 \\ \frac{1}{M} \\ 0 \\ -\frac{1}{L_0 M} \end{pmatrix} \end{aligned} \tag{17}$$

The new state vectors ξ_1 and ξ_2 are the following:

$$\xi_1 = \begin{pmatrix} l \\ \dot{l} \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} r \\ \dot{r} \\ \theta \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} \tag{18}$$