Computer Controlled Systems
Homework 3
Submission deadline: November 16. 2017. 10:00/12:00 (end of the seminar)

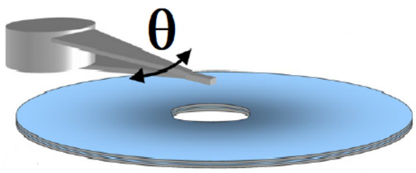
All solutions are expected to be calculated by hand, also all figures have to be drawn by hand. Computer programs can be used for self-verification, but all problems have to contain the detailed steps of solutions!

Frequency response analysis and controller design for a HDD head[1]

Consider a HDD read/write head with dynamics described by the following differential equation:

$$J\ddot{\theta} + b\dot{\theta} + k_1\theta = k_2 u$$

where θ denotes the angular position of the head, while u is the input current.



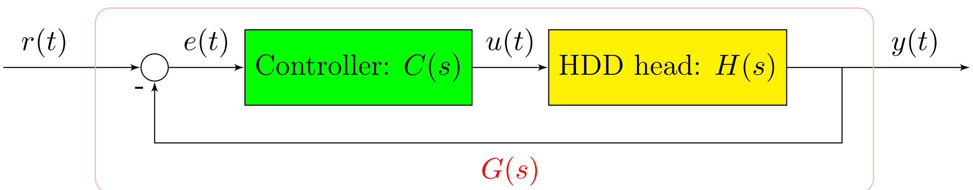
The names and values of the system constants are:

$J = 0.01\ kgm^2$	inertia of the head assembly
$b = 0.004\ Nm/(rad/sec)$	viscous damping coefficient of the bearings
$k_1 = 10\ Nm/rad$	return spring constant
$k_2 = 0.05\ Nm/rad$	motor torque constant

By choosing the angular position as output variable ($y = \theta$), the input-output model of the HDD head is:

$$\ddot{y}(t) + 0.4\dot{y}(t) + 1000y(t) = 5u(t)$$

The head has a (damped) oscillatory behaviour, which is unacceptable. Moreover, it has to be positioned to prescribed values in a short time. To fulfill these needs, a PID controller is applied to the system as depicted in the figure below.



1. Determine the transfer function $H(s)$ of the input-output model by Laplace-transform!

   Solution.
   $$s^2Y(s) + 0.4sY(s) + 1000Y(s) = 5U(s) \quad \Rightarrow \quad H(s) = \frac{5}{s^2 + 0.4s + 1000}$$

2. Check the poles of $H(s)$! Is the system stable? What refers to the oscillatory behaviour?

   *Solution.*The poles are the roots of the denominator: $\lambda_{1,2} = -0.2 \pm 31.622j$. Since $Re(\lambda_{1,2}) < 0$ the system is stable. The complex conjugate poles refer to the oscillatory behaviour.

3. Give the frequency response function $H(j\omega)$ of the system! Give its gain $k(\omega)$ and phase $\phi(\omega)$ at $\omega = 0\ rad/s$ and $\omega = 32\ rad/s$! What is the response $y(t)$ to the periodic input $u(t) = 10sin(32t)$?

[1] Model is borrowed from MathWorks

Solution.

$$H(j\omega) = \frac{5}{(j\omega)^2 + 0.4(j\omega) + 1000}$$

At $\omega = 0$:

$$H(j0) = \frac{5}{1000}, \quad k = |H(j0)| = 0.005, \quad \phi = arctg(\frac{Im(H(j0))}{Re(H(j0))}) = 0$$

At $\omega = 32$:

$$\begin{aligned}
H(j32) &= \frac{5}{(j32)^2 + 0.4(j32) + 1000} = \frac{5}{-24 + j12.8} \\
k &= |H(j32)| = \frac{5}{\sqrt{(-24)^2 + 12.8^2}} = 0.184 \\
H(j32) &= \frac{5}{-24 + j12.8} \cdot \frac{-24 - j12.8}{-24 - j12.8} = \frac{-120 - j64}{(-24)^2 - (j12.8)^2} = \frac{-120}{24^2 + 12.8^2} + j\frac{-64}{24^2 + 12.8^2} = \\
&= -0.1622 - 0.0865j \\
\phi &= arctg(\frac{Im(H(j32))}{Re(H(j32))}) = arctg(\frac{-0.0865}{-0.1622}) = 0.49\ rad
\end{aligned}$$

The system response is:

$$y(t) = k \cdot 10sin(32t + \phi) = 1.84sin(32t + 0.49)$$

4. Give the transfer function $G(s)$ of the closed loop system!

Solution.

$$\begin{aligned}
G(s) &= \frac{K_{PID}(s)H(s)}{1 + K_{PID}(s)H(s)} = \frac{(K_P + K_D s + K_I \frac{1}{s})\frac{5}{s^2+0.4s+1000}}{1 + (K_P + K_D s + K_I \frac{1}{s})\frac{5}{s^2+0.4s+1000}} = \\
&= \frac{5(K_P + K_D s + K_I \frac{1}{s})}{s^2 + 0.4 + 1000 + 5(K_P + K_D s + K_I \frac{1}{s})} = \frac{5(K_D s^2 + K_P s + K_I)}{s^3 + (5K_D + 0.4)s^2 + (5K_P + 1000)s + 5K_I}
\end{aligned}$$

5. Determine the tuning parameters K_P, K_D and K_I in such a way that all the poles of the closed loop system are equal to -20 !

Solution. The denominator of $G(s)$ is a third order polynomial therefore it has three poles. We have to choose K_P, K_D and K_I in such a way that the denominator is equal to $(s - 20)^3$:

$$s^3 + (5K_D + 0.4)s^2 + (5K_P + 1000)s + 5K_I = s^3 + 60s^2 + 1200s + 8000$$

This determines the values of the tuning parameters:

$$K_P = 40,\ K_D = 11.92,\ K_I = 1600$$

6. Determine the DC gain of the closed loop system! Does this controlled system follow a constant reference (head positioning) signal?

Solution. DC gain is $G(j0) = 1$ (this is the effect of the integrator). Yes, it follows because of the integrator.