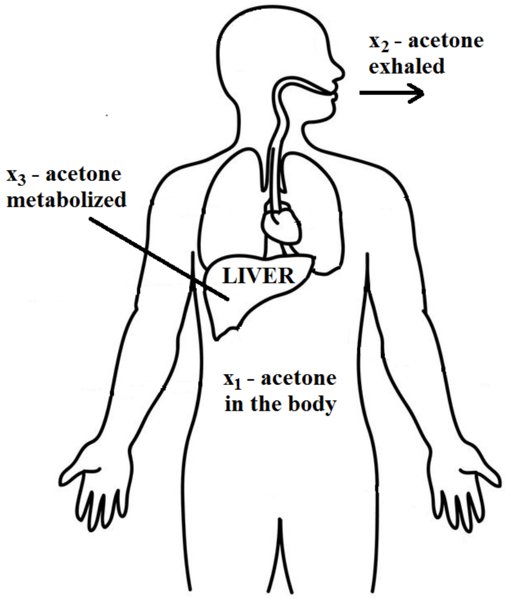
Computer Controlled Systems
Homework 2
Submission deadline: October 26. 2017. 10:00/12:00 (end of the seminar)

All solutions are expected to be calculated by hand, also all figures have to be drawn by hand. Computer programs can be used for self-verification, but all problems have to contain the detailed steps of solutions!

I. Kinetics of acetone in the human body



The kinetics of the distribution of acetone in the human body is described by the following differential equations:

$$\begin{aligned} \dot{x}_1(t) &= -K_{exh} \cdot x_1(t) - K_{met} \cdot x_1(t) + K_{in} \cdot u(t), & x_1(0) &= x_{1_0} \\ \dot{x}_2(t) &= K_{exh} \cdot x_1(t), & x_2(0) &= 0 \\ \dot{x}_3(t) &= K_{met} \cdot x_1(t), & x_3(0) &= 0 \end{aligned}$$

where x_1 $[g]$ is the amount of acetone in the body, x_2 $[g]$ and x_3 $[g]$ are the amounts of acetone exhaled and metabolized, respectively, while u $\left[\frac{g}{h}\right]$ is the inlet rate of acetone to the body. The system parameters are:

$K_{exh} = 4\frac{1}{h}$	exhalation rate constant
$K_{met} = 2\frac{1}{h}$	metabolic rate constant
$K_{in} = 1$	body inlet rate constant

By choosing the - easily measurable - amount of exhaled acetone as output: $y(t) = x_2(t)$ and $u(t)$ as input, with the state variable $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}$ a third order state-space model is derived:

$$\begin{aligned} \dot{x}_1(t) &= -6x_1(t) + u(t) \\ \dot{x}_2(t) &= 4x_1(t) \\ \dot{x}_3(t) &= 2x_1(t) \\ y(t) &= x_2(t) \end{aligned}$$

with the initial conditions $x_1(0) = x_{1_0}$, $x_2(0) = 0$ and $x_3(0) = 0$.

Problems

1. Is this state space model controllable/observable?

   Solution. This state space model can be written in matrix-vector form:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \quad , \quad A = \begin{bmatrix} -6 & 0 & 0 \\ 4 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \; C = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$$

   The controllability matrix is:

$$\mathcal{C}_3 = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & -6 & 36 \\ 0 & 4 & -24 \\ 0 & 2 & -12 \end{bmatrix} .$$

   Since $\det(\mathcal{C}_3) = 0$, $\mathcal{C}_3$ is not of full rank, therefore the state space model is NOT controllable.

   The observability matrix is:

$$\mathcal{O}_3 = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 4 & 0 & 0 \\ -24 & 0 & 0 \end{bmatrix} .$$

Since $\det(\mathcal{O}_3) = 0$, $\mathcal{O}_3$ is not of full rank, therefore the state space model is NOT observable.

2. If applicable, determine the controllable/unobservable subspaces of the state space.

   Solution. Since the second and third columns of $\mathcal{C}_3$ are linearly dependent, the controllable subspace is

$$im(\mathcal{C}_3) = span\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}\}$$

   The unobservable subspace is the kernel of $\mathcal{O}_3$, and it is the solution of the linear set of equations $\mathcal{O}_3 \cdot x = 0$:

$$ker(\mathcal{O}_3) = span\{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\} .$$

3. Give a minimal state space realization for this system.

   First we determine the transfer function:

$$\begin{aligned} H(s) &= C(sI - A)^{-1}B = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} s+6 & 0 & 0 \\ -4 & s & 0 \\ -2 & 0 & s \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \\ &= \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{s+6} & 0 & 0 \\ \frac{4}{s(s+6)} & \frac{1}{s} & 0 \\ \frac{2}{s(s+6)} & 0 & \frac{1}{s} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{4}{s(s+6)} = \frac{4}{s^2+6s} \end{aligned}$$

   where the inverse of $(sI - A)$ has been determined by Gauss-Jordan elimination. Observe that $H(s)$ is of second order meaning that it is a reduced transfer function (it is of third order originally since we have 3 state variables).

   We can give different minimal realizations:

   Controller form realization:

$$\begin{aligned} \dot{x}_c(t) &= \begin{bmatrix} -6 & 0 \\ 1 & 0 \end{bmatrix} x_c(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 0 & 4 \end{bmatrix} x_c(t) \end{aligned}$$

   Observer form realization:

$$\begin{aligned} \dot{x}_o(t) &= \begin{bmatrix} -6 & 1 \\ 0 & 0 \end{bmatrix} x_o(t) + \begin{bmatrix} 0 \\ 4 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x_o(t) \end{aligned}$$

   One of the diagonal realizations:

$$H(s) = \frac{4}{s(s+6)} = \frac{\frac{4}{6}}{s} - \frac{\frac{4}{6}}{s+6} \quad \Longrightarrow \quad \begin{aligned} \dot{x}_d(t) &= \begin{bmatrix} 0 & 0 \\ 0 & -6 \end{bmatrix} x_d(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} \frac{4}{6} & -\frac{4}{6} \end{bmatrix} x_d(t) \end{aligned}$$

   A minimal realization can be also given by simply cutting off x_3 from the state space model, since it has no effect neither on the output nor on the other state variables ($\dot{x}_1$ and $\dot{x}_2$ does not depend on x_3):

$$\begin{aligned} \dot{x}_r(t) &= \begin{bmatrix} -6 & 0 \\ 4 & 0 \end{bmatrix} x_r(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 0 & 1 \end{bmatrix} x_r(t) \end{aligned}$$

4. Determine whether the system BIBO stable or not?

   Solution.

$$H(s) = \frac{\frac{4}{6}}{s} - \frac{\frac{4}{6}}{s+6} \quad \Longrightarrow \quad h(t) = \mathcal{L}^{-1}\{H\}(t) = \frac{4}{6}(1 - e^{-6t})$$

We check the absolute integrability of h:

$$\begin{aligned}\int_0^\infty |h(t)|dt &= \int_0^\infty |\frac{4}{6}(1-e^{-6t})|dt = \frac{4}{6}\int_0^\infty |1-e^{-6t}|dt = \frac{4}{6}\int_0^\infty 1-e^{-6t}dt = \\ &= \lim_{T\to\infty}\frac{4}{6}\left[t+\frac{1}{6}e^{-6t}\right]_{t=0}^{t=T} = \infty\end{aligned}$$

therefore the system is not BIBO stable.

II. Stability analysis of a nonlinear system

The motion of a planar pendulum is described by the following differential equation system

$$ml\ddot{\theta} + mg\sin\theta = 0 \quad\Longleftrightarrow\quad \ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

where m denotes the mass of the body, l denotes the length of the pendulum, g is the gravitational acceleration. Variable θ gives the angle of the pendulum relatively to the vertical. In order to build a state space model, we introduce the following state variables:

$$\begin{aligned} x_1 &= \theta \\ x_2 &= \dot{\theta} \end{aligned}$$

Than the state space model of the planar pendulum is the following:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{l}\sin x_1 \end{aligned}$$

1. Show that $x^* = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$ is an equilibrium point of the system!

   Solution. Since $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$ if $x_1 = 0$ and $x_2 = 0$, the origin is an equilibrium point.

2. Consider the following Lyapunov function candidate

   $$V(x) = mgl(1-\cos x_1) + \frac{1}{2}ml^2x_2^2, \tag{1}$$

   that is the pendulum total energy. Using $V(x)$ analyse the stability properties of this equilibrium point. Is it stable? Is it asymptotically stable?

   Solution. V is positive everywhere (except the origin, where it is zero). Its time derivative is

   $$\begin{aligned}\dot{V}(x) &= mgl\dot{x}_1\sin x_1 + ml^2x_2\dot{x}_2 \\ &= mglx_2\sin x_1 + ml^2x_2(-\frac{g}{l}\sin x_1) = 0\end{aligned}$$

   Consequently, the system is (globally) stable, but not in asymptotic sense. This can be explained by the conservativeness of the system since there is no loss of energy.

3. Assuming friction $F = bl\dot{\theta}$, we dynamics will look like

   $$ml\ddot{\theta} + bl\dot{\theta} + mg\sin\theta = 0,$$

   where $b > 0$ is the damping factor.

   Hence, the state space equations become

   $$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{l}\sin x_1 - \frac{b}{m}x_2 \end{aligned}$$

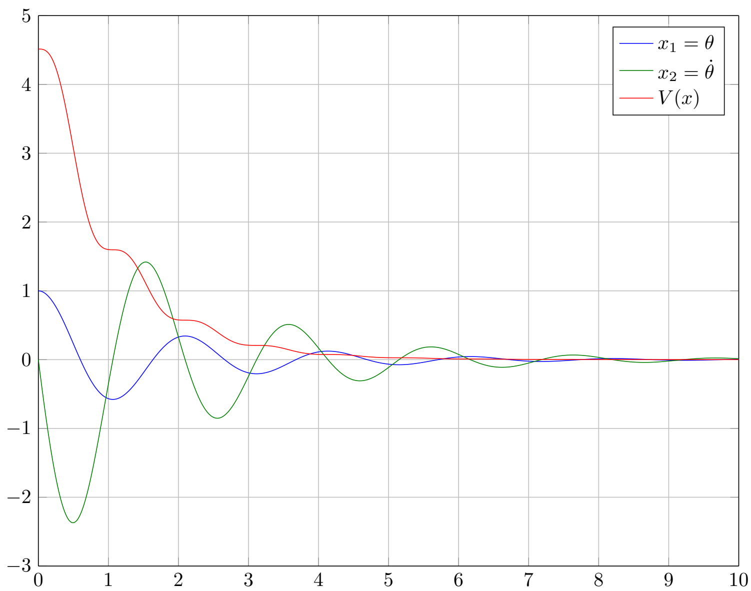
   Using the same function $V(x)$ analyse the stability properties of this equilibrium point. Is it stable? Is it asymptotically stable?

Solution. $V(x)$ is globally positive outside the origin, and $V(0) = 0$. Its time derivative is:

$$\begin{aligned}\dot{V}(x) &= mgl\dot{x}_1 \sin x_1 + ml^2 x_2 \dot{x}_2 \\ &= mglx_2 \sin x_1 + ml^2 x_2(-\frac{g}{l}\sin x_1 - \frac{b}{ml^2}x_2) = -bx_2^2 < 0, \quad \forall x, \text{ where } x_2 \neq 0\end{aligned}$$

According to the Theorem learnt, the equilibrium point is globally stable - however, its asymptotic stability can also be shown.

A remark about asymptotic stability:
Notice that $\dot{V}(x) = 0$ for any $x = [x_1, 0]^T$, nevertheless the system can be *asymptotically* stable. The following simulation demonstrates a case where the initial condition is $x_0 = [1, 0]^T$. We can see, however, that the value of the angular velocity (x_2) is mostly nonzero during the simulation (green curve), therefore the energy of the system (the value of the Lyapunov function along the trajectory – red curve) decreases as time goes by. One can also consider that in those places where the angular velocity (x_2) is equal to zero, the time derivative of the Lyapunov function along the trajectory is equal to zero (the derivative of the "red function" is zero there), which means that the system keeps its energy for a moment. This occurs in those cases when the pendulum has reached its maximal deviation but it has not started to fall yet.



The numerical parameters of the simulation are: $g = 9.82[m/s^2]$, $l = 1[m]$, $m = 1[kg]$, $b = 1[kg\, m^2/s]$

III. Kalman decomposition

Compulsory only for the students of the course "01TG" (csak a tehetséggondozásban résztvevő hallgatók számára kötelező).

Problem 1. Given a strictly proper ($D = 0$) state space model (A, B, C). Based on the appendices of the lecture notes, prove that $v \in \mathrm{Im}(\mathcal{O}_n^T)$ implies $A^T v \in \mathrm{Im}(\mathcal{O}_n^T)$, where $\mathcal{O}_n$ is the observability matrix.

Problem 2. Given a redundant 8th order SISO state space model ($A \in \mathbb{R}^{8\times 8}, B, C, D$). The values of the model matrices are given in the binary kalman_decomp_ABCD.mat file. Produce the Kalman decomposition of this system. You can use the following build-in Matlab functions: `svd`, `null`, `orth`, `rank`. In order to check your computations, you may use `ss`, `tf`, `minreal`.