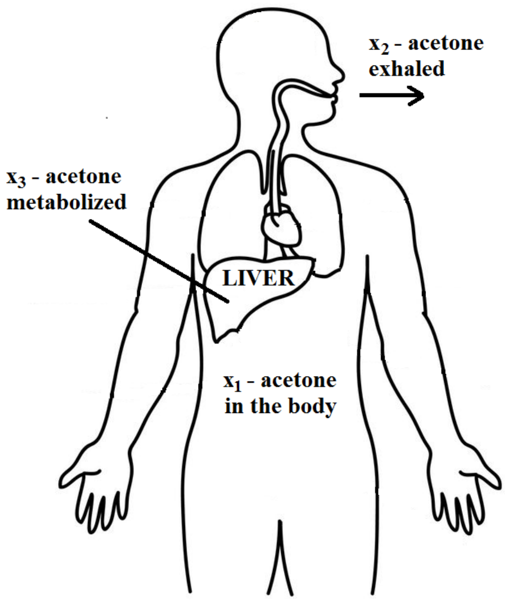
Computer Controlled Systems
Homework 2
Submission deadline: October 26. 2017. 10:00/12:00 (end of the seminar)

All solutions are expected to be calculated by hand, also all figures have to be drawn by hand. Computer programs can be used for self-verification, but all problems have to contain the detailed steps of solutions!

I. Kinetics of acetone in the human body



The kinetics of the distribution of acetone in the human body is described by the following differential equations:

$$\begin{aligned} \dot{x}_1(t) &= -K_{exh} \cdot x_1(t) - K_{met} \cdot x_1(t) + K_{in} \cdot u(t), & x_1(0) &= x_{1_0} \\ \dot{x}_2(t) &= K_{exh} \cdot x_1(t), & x_2(0) &= 0 \\ \dot{x}_3(t) &= K_{met} \cdot x_1(t), & x_3(0) &= 0 \end{aligned}$$

where x_1 $[g]$ is the amount of acetone in the body, x_2 $[g]$ and x_3 $[g]$ are the amounts of acetone exhaled and metabolized, respectively, while $u\left[\frac{g}{h}\right]$ is the inlet rate of acetone to the body. The system parameters are:

$K_{exh} = 4\frac{1}{h}$	exhalation rate constant
$K_{met} = 2\frac{1}{h}$	metabolic rate constant
$K_{in} = 1$	body inlet rate constant

By choosing the - easily measurable - amount of exhaled acetone as output: $y(t) = x_2(t)$ and $u(t)$ as input, with the state variable $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}$ a third order state-space model is derived:

$$\begin{aligned} \dot{x}_1(t) &= -6x_1(t) + u(t) \\ \dot{x}_2(t) &= 4x_1(t) \\ \dot{x}_3(t) &= 2x_1(t) \\ y(t) &= x_2(t) \end{aligned}$$

with the initial conditions $x_1(0) = x_{1_0}$, $x_2(0) = 0$ and $x_3(0) = 0$.

Problems

1. Is this state space model controllable/observable?
2. If applicable, determine the controllable/unobservable subspaces of the state space.
3. Give a minimal state space realization for this system.
4. Determine whether the system BIBO stable or not?

II. Stability analysis of a nonlinear system

The motion of a planar pendulum is described by the following differential equation system

$$ml\ddot{\theta} + mg\sin\theta = 0 \quad \Longleftrightarrow \quad \ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

where m denotes the mass of the body, l denotes the length of the pendulum, g is the gravitational acceleration. Variable θ gives the angle of the pendulum relatively to the vertical. In order to build a state space model, we introduce the following state variables:

$$\begin{aligned} x_1 &= \theta \\ x_2 &= \dot{\theta} \end{aligned}$$

Than the state space model of the planar pendulum is the following:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{l}\sin x_1 \end{aligned}$$

1. Show that $x^* = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$ is an equilibrium point of the system!
2. Consider the following Lyapunov function candidate

   $$V(x) = mgl(1 - \cos x_1) + \frac{1}{2}ml^2x_2^2, \tag{1}$$

   that is the pendulum total energy. Using $V(x)$ analyse the stability properties of this equilibrium point. Is it stable? Is it asymptotically stable?
3. Assuming friction $F = bl\dot{\theta}$, we dynamics will look like

   $$ml\ddot{\theta} + bl\dot{\theta} + mg\sin\theta = 0,$$

   where $b > 0$ is the damping factor.

   Hence, the state space equations become

   $$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{l}\sin x_1 - \frac{b}{m}x_2 \end{aligned}$$

   Using the same function $V(x)$ analyse the stability properties of this equilibrium point. Is it stable? Is it asymptotically stable?

III. Kalman decomposition

Compulsory only for the students of the course “01TG” (csak a tehetséggondozásban résztvevő hallgatók számára kötelező).

Problem 1. Given a strictly proper ($D = 0$) state space model (A, B, C). Based on the appendices of the lecture notes, prove that $v \in \mathrm{Im}(\mathcal{O}_n^T)$ implies $A^Tv \in \mathrm{Im}(\mathcal{O}_n^T)$, where $\mathcal{O}_n$ is the observability matrix.

Problem 2. Given a redundant 8th order SISO state space model ($A \in \mathbb{R}^{8\times 8}, B, C, D$). The values of the model matrices are given in the binary kalman_decomp_ABCD.mat file. Produce the Kalman decomposition of this system. You can use the following build-in Matlab functions: `svd`, `null`, `orth`, `rank`. In order to check your computations, you may use `ss`, `tf`, `minreal`.