**Computer Controlled Systems**
Homework 1
Submission deadline: October 5. 2017. 10:00/12:00 (end of the seminar)

All solutions are expected to be calculated by hand, also all figures have to be drawn by hand. Computer programs can be used for self-verification, but all problems have to contain the detailed steps of solutions!

**Problem 1.** Given a linear mapping $\mathcal{A} : \mathbb{R}^3 \to \mathbb{R}^3$ by the matrix

$$A = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -2 & 0 \\ 2 & 1 & -3 \end{pmatrix} \tag{1}$$

1. Apply this linear mapping to the vector $w = \left(\begin{smallmatrix} w_1 \\ w_2 \\ w_3 \end{smallmatrix}\right) \in \mathbb{R}^3$.

   *Solution:* $\bar{w} = Aw = \begin{pmatrix} w_2 - w_1 \\ -2w_2 \\ 2w_1 + w_2 - 3w_3 \end{pmatrix}$

2. Calculate the eigenvalues and the corresponding eigenvectors of matrix $A$.
   (The eigenvalues may be complex as well.)
   *Solution:*
   The eigenvalues are: $\lambda_1 = 1$, $\lambda_2 =$, $\lambda_3 = -1$
   Corresponding eigenvectors:

   $$s_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad s_3 = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$$

3. Determine the diagonal matrix $D$ of the mapping $\mathcal{A}$ and give an appropriate transformation matrix $S$ such that $D = S^{-1} \cdot A \cdot S$.
   *Solution:*
   $D$ is a diagonal matrix containing the eigenvalues calculated. The columns of $S$ are the corresponding eigenvectors with $p_1 = p_2 = p_3 = 1$, while $S^{-1}$ is the inverse of $S$:

   $$D = \begin{pmatrix} -3 & & \\ & -1 & \\ & & -2 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix}, \quad S^{-1} = \begin{pmatrix} -1 & -0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{2}$$

   Note that in the construction of $S$, arbitrary values for $p_1, p_2, p_3$ can be chosen. For the sake of simplicity we used $p_1 = p_2 = p_3 = 1$.

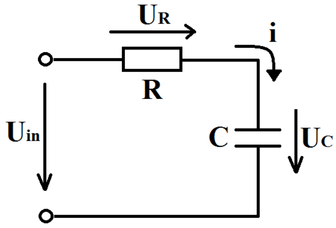
4. Give the exponential matrix $e^D$.
   *Solution:*
   Since $D$ is a diagonal matrix, $D^n = \begin{pmatrix} (-3)^n & & \\ & (-1)^n & \\ & & (-2)^n \end{pmatrix}$, and its exponential matrix can be written as

   $$e^D = \sum_{n=0}^{\infty} \frac{A^n}{n!} = \begin{pmatrix} \sum_{n=0}^{\infty} \frac{(-3)^n}{n!} & 0 & 0 \\ 0 & \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} & 0 \\ 0 & 0 & \sum_{n=0}^{\infty} \frac{(-2)^n}{n!} \end{pmatrix} = \begin{pmatrix} e^{-3} & 0 & 0 \\ 0 & e^{-1} & 0 \\ 0 & 0 & e^{-2} \end{pmatrix}$$

5. Determine the exponential matrix $e^A$.

   *Solution:*

   $$e^A = Se^DS^{-1} = \begin{pmatrix} e^{-1} & e^{-1} - e^{-2} & 0 \\ 0 & e^{-2} & 0 \\ e^{-1} - e^{-3} & e^{-1} - e^{-2} & e^{-3} \end{pmatrix} \tag{3}$$




**Problem 2.** Consider a simple electric circuit with a resistor and a capacitor!

The capacitor and the resistor are ideal elements with constant resistance and capacitance: $R = 1k\Omega$, $C = 1mF$. $U_{in}$ is an external voltage, $i$ is the current of the circuit. Note that $U_C = 0$ at $t = 0$.
Using Kirchoff's second law (the sum of voltages is equal to zero, i.e. $U_{in} - R \cdot i - U_C = 0$), and the differential equation for the capacitor $i = C \cdot \dot{U}_C$, a first order differential equation with constant coefficients can be derived as:

$$\dot{U}_C + \frac{1}{RC}U_C = \frac{1}{RC}U_{in} \ , \quad U_C(0) = 0$$

By choosing the input as $u(t) = U_{in}(t)$ and the output as $y(t) = U_C(t)$ and substituting the values of $R$ and $C$, we get the input-output model of the circuit:

$$\dot{y}(t) + y(t) = u(t) \ , \quad y(0) = 0$$

1. Determine the transfer function $H(s) = \frac{Y(s)}{U(s)}$ of the system with Laplace-transform!

   *Solution:*
   Applying Laplace-transform:

   $$\mathcal{L}\{\dot{y}(t)\} + \mathcal{L}\{y(t)\} = \mathcal{L}\{u(t)\}$$

   and using that $\mathcal{L}\{\dot{y}(t)\} = s\mathcal{L}\{y(t)\} - y(0) = sY(s)$, the L-transformed equation is:

   $$s \cdot Y(s) + Y(s) = U(s)$$

   From this, H(s) can be determined:

   $$(s+1) \cdot Y(s) = U(s) \quad \Rightarrow \quad H(s) = \frac{Y(s)}{U(s)} = \frac{1}{s+1}$$

2. Determine the impulse-response function $h(t)$ !

   *Solution:*
   By inverse Laplace-transform:

   $$h(t) = \mathcal{L}^{-1}\{H(s)\} = e^{-t}$$

3. Give the time-response $y(t)$ for the unit step function as input: $u(t) = 1(t)$ *by convolution!*

   **In order to check your solution**, the impulse and step response of a system given by its transfer function can be easily computed using Matlab. Eg. $H(s) = \frac{12s^2+1}{s^3+1}$

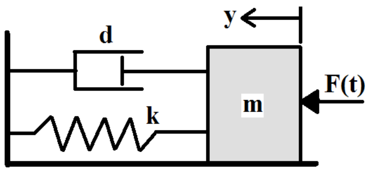
```
s = tf('s');
H = ( 12 * s^2 + 1 ) / ( s^3 + 1 );
impulse(H) % plot the impulse response of H
step(H)    % plot the step response of H
```

   *Solution:*

   $$y(t) = \int_0^t h(\tau)\mathrm{d}\tau = \int_0^t e^{-\tau}\mathrm{d}\tau = \left[-e^{-\tau}\right]_{\tau=0}^{\tau=t} = 1 - e^{-t}$$

**Problem 3.** Consider a simple spring-mass system with damping and an external force $F(t)$. The time-dependent position of the mass is $y(t)$, where the zero position $y = 0$ belongs to the relaxed state of the spring. The damping is of viscous type: it is proportional to $\dot{y}$ which is the velocity of the mass.

The system constants are the spring constant $k = 4\frac{N}{m}$, the damping coefficient $d = 5\frac{kg}{s}$, and $m = 1kg$. When $t = 0$, the mass is at position 0, in other words, the initial condition constitutes $y(0) = 0$.




The differential equation describing the dynamics of the spring-mass system is

$$\ddot{y}(t) + \frac{d}{m}\dot{y}(t) + \frac{k}{m}y(t) = \frac{1}{m}F(t), \quad \text{with initial conditions: } y(0),\ \dot{y}(0). \tag{4}$$

Using the values of system constants, and applying an external $F(t) = -2e^{-3t} - 4e^{-5t}N$, the differential equation of the spring-mass system is as follows:

$$\ddot{y}(t) + 5\dot{y}(t) + 4y(t) = -2e^{-3t} - 4e^{-5t}, \quad y(0) = 0, \quad \dot{y}(0) = 5 \tag{5}$$

1. Using Laplace-transform, determine $y(t)$, the time function of the position of the mass.

   *Solution.* Applying Laplace-transform:

   $$4Y(s) + 5sY(s) + s^2Y(s) - 5 = -\frac{2}{s+3} - \frac{4}{s+5} \Rightarrow (s^2 + 5s + 4)Y(s) = \frac{5s^2 + 34s + 53}{(s+3)(s+5)} \tag{6}$$

   Partial fractional decomposition:

   $$Y(s) = \frac{5s^2 + 34s + 53}{(s+1)(s+3)(s+4)(s+5)} = \frac{C_1}{s+1} + \frac{C_2}{s+3} + \frac{C_3}{s+4} + \frac{C_4}{s+5} \tag{7}$$

   Since all roots of the denominator are different, the following limits give the coefficients of the partial fractions:

   $$\begin{aligned} C_1 &= \lim_{s\to -1} \frac{5s^2 + 34s + 53}{(s+3)(s+4)(s+5)} = 1, \quad & C_2 &= \lim_{s\to -3} \frac{5s^2 + 34s + 53}{(s+1)(s+4)(s+5)} = 1 \\ C_3 &= \lim_{s\to -4} \frac{5s^2 + 34s + 53}{(s+1)(s+3)(s+5)} = -1, \quad & C_3 &= \lim_{s\to -5} \frac{5s^2 + 34s + 53}{(s+1)(s+3)(s+4)} = -1 \end{aligned} \tag{8}$$

   Thus

   $$Y(s) = \frac{5s^2 + 34s + 53}{(s+1)(s+3)(s+4)(s+5)} = \frac{1}{s+1} + \frac{1}{s+3} - \frac{1}{s+4} - \frac{1}{s+5} \tag{9}$$

   The solution is given by inverse Laplace-transform:

   $$y(t) = e^{-t} + e^{-3t} - e^{-4t} - e^{-5t} \tag{10}$$

2. Is there any oscillation in the movement of the mass?

   *Solution:*
   Since all the terms of $y(t)$ are in the form of $e^{-at}$, where $a \in \mathbb{R}^+$, there is no oscillation in $y(t)$.

3. Determine $\lim_{t\to\infty} y(t)$.

   *Solution:*
   $\lim_{t\to\infty} y(t) = 0$ since $\lim_{t\to\infty} e^{-at} = 0$ , $\forall a \in \mathbb{R}^+$ .

4. Using Laplace transform, determine the transfer function ($H(s)$) of the system if $u(t) := F(t)$ is considered as an input signal and $y(t)$ is the output signal.
   *Solution.* We have to determine the transfer function of (4), where $u(t) = F(t)$ is the input and $y(t)$ is the output. Substituting the parameters $m$, $k$, $d$, we obtain:

   $$\ddot{y}(t) + 5\dot{y}(t) + 4y(t) = u(t). \tag{11}$$

   Thus, the transfer function is: $H(s) = \frac{0s+1}{s^2+5s+4}$.

5. Using $H(s)$, give a controllable state-space realization of the system.
   *Solution.*

$$A = \begin{pmatrix} -5 & -4 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 \end{pmatrix}, \quad D = 0. \tag{12}$$

Compulsory only for the studends of the course "01TG" (csak a tehetséggondozásban résztvevő hallgatók számára kötelező).

**Problem 4.** (Basic numeric operations in system's theory)

We consider a strictly proper ($D = 0$) 3-dimensional SISO state-space model

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx, \end{aligned} \qquad A = \begin{pmatrix} -2 & -3 & -8 \\ 0 & -3 & -6 \\ 0 & 1 & -3 \end{pmatrix}, \quad B = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}. \tag{13}$$

1. Compute the diagonal matrix $D$ and invertible matrix $S$ such that $A = SDS^{-1}$. Use the function `eig`.
2. Solve the state-space model using the numeric built-in solver `ode45` if
   (a) $x(0) \neq 0$, $u(t) = 0$. Try different values for the initial conditions $x(0)$, and plot the resulting trajectories $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}$ alongside each other using a 3D plot (`plot3`). See Figure 1.
   (b) $x(0) = 0$, $u(t) = \sin(\omega t)$. Try different values for $\omega$.

   Additionally, in both cases, plot the state variables $x_i(t)$ in the function of the time.
3. Plot the impulse response $h(t)$ of the system (`impulse`).
4. Determine the transfer function $H(s)$ of the system from $u$ to $y$. Give the coefficients of the numerator $b(s)$ and denominator $a(s)$ of the transfer function $H(s)\frac{b(s)}{a(s)}$. You can use functions `ss, tf, tfdata` or `ss2tf` (deprecated).

*Solution.* published Matlab script.

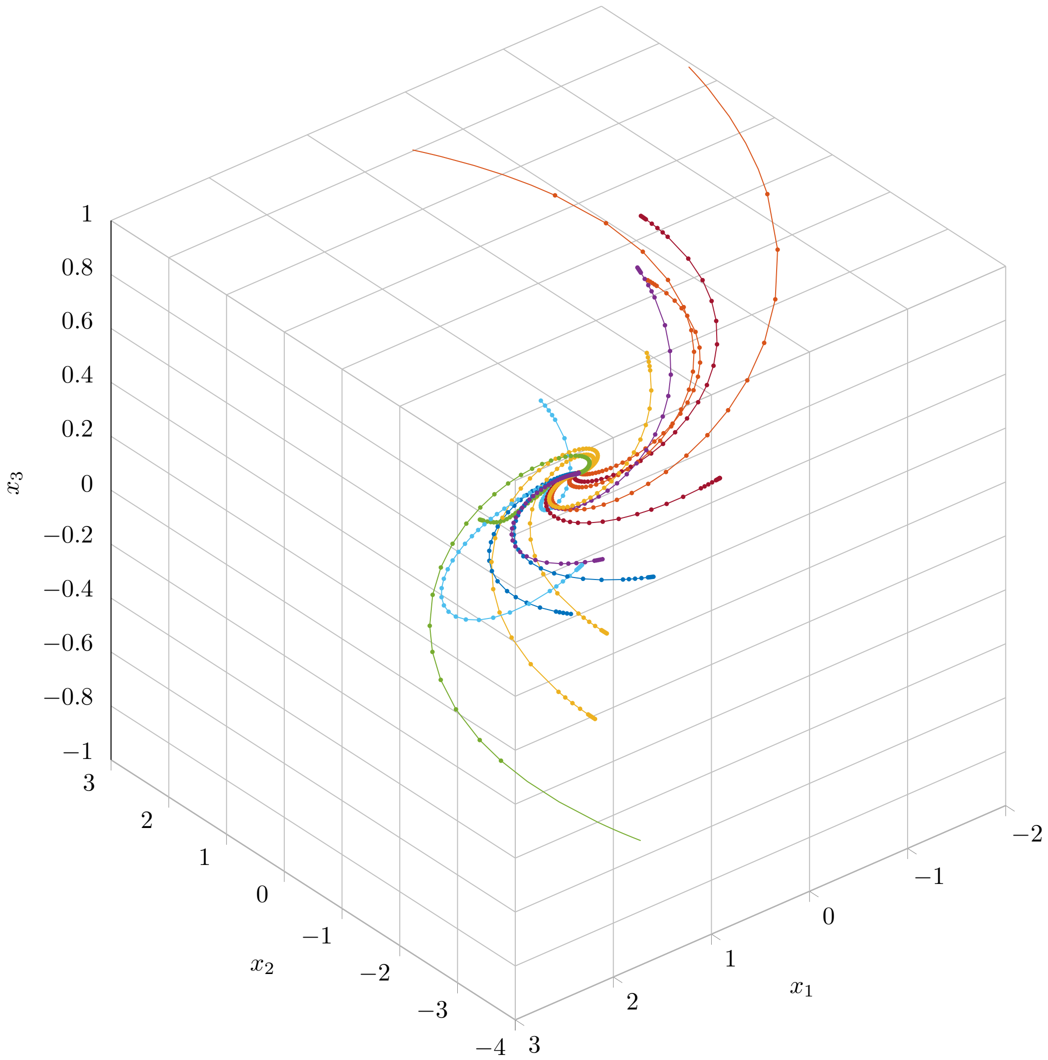



Figure 1. Phase portrait of the 3-dimensional system given in Problem 4.