Computer Controlled Systems
Homework 1
Submission deadline: October 5. 2017. 10:00/12:00 (end of the seminar)

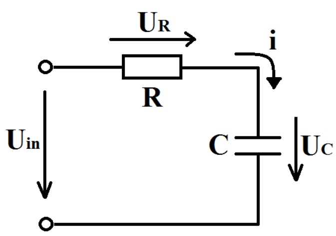
All solutions are expected to be calculated by hand, also all figures have to be drawn by hand. Computer programs can be used for self-verification, but all problems have to contain the detailed steps of solutions!

Problem 1. Given a linear mapping $\mathcal{A} : \mathbb{R}^3 \to \mathbb{R}^3$ by the matrix

$$A = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -2 & 0 \\ 2 & 1 & -3 \end{pmatrix} \tag{1}$$

1. Apply this linear mapping to the vector $w = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \in \mathbb{R}^3$.
2. Calculate the eigenvalues and the corresponding eigenvectors of matrix A. (The eigenvalues may be complex as well.)
3. Determine the diagonal matrix D of the mapping $\mathcal{A}$ and give an appropriate transformation matrix S such that $D = S^{-1} \cdot A \cdot S$.
4. Give the exponential matrix e^D.
5. Determine the exponential matrix e^A.

Problem 2. Consider a simple electric circuit with a resistor and a capacitor!



The capacitor and the resistor are ideal elements with constant resistance and capacitance: $R = 1k\Omega$, $C = 1mF$. U_{in} is an external voltage, i is the current of the circuit. Note that $U_C = 0$ at $t = 0$.
Using Kirchoff's second law (the sum of voltages is equal to zero, i.e. $U_{in} - R \cdot i - U_C = 0$), and the differential equation for the capacitor $i = C \cdot \dot{U}_C$, a first order differential equation with constant coefficients can be derived as:

$$\dot{U}_C + \frac{1}{RC} U_C = \frac{1}{RC} U_{in} \ , \quad U_C(0) = 0$$

By choosing the input as $u(t) = U_{in}(t)$ and the output as $y(t) = U_C(t)$ and substituting the values of R and C, we get the input-output model of the circuit:

$$\dot{y}(t) + y(t) = u(t) \ , \quad y(0) = 0$$

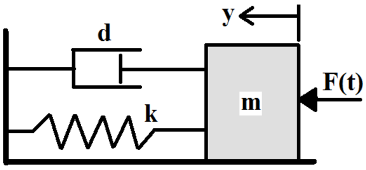
1. Determine the transfer function $H(s) = \frac{Y(s)}{U(s)}$ of the system with Laplace-transform!
2. Determine the impulse-response function $h(t)$!
3. Give the time-response $y(t)$ for the unit step function as input: $u(t) = 1(t)$ *by convolution!*

   In order to check your solution, the impulse and step response of a system given by its transfer function can be easily computed using Matlab. Eg. $H(s) = \frac{12s^2+1}{s^3+1}$

```
s = tf('s');
H = ( 12 * s^2 + 1 ) / ( s^3 + 1 );
impulse(H) % plot the impulse response of H
```

```
step(H)    % plot the step response of H
```

Problem 3. Consider a simple spring-mass system with damping and an external force $F(t)$. The time-dependent position of the mass is $y(t)$, where the zero position $y = 0$ belongs to the relaxed state of the spring. The damping is of viscous type: it is proportional to $\dot{y}$ which is the velocity of the mass. The system constants are the spring constant $k = 4\frac{N}{m}$, the damping coefficient $d = 5\frac{kg}{s}$, and $m = 1kg$. When $t = 0$, the mass is at position 0, in other words, the initial condition constitutes $y(0) = 0$.



The differential equation describing the dynamics of the spring-mass system is

$$\ddot{y}(t) + \frac{d}{m}\dot{y}(t) + \frac{k}{m}y(t) = \frac{1}{m}F(t), \quad \text{with initial conditions:} \quad y(0),\ \dot{y}(0). \tag{2}$$

Using the values of system constants, and applying an external $F(t) = -2e^{-3t} - 4e^{-5t}N$, the differential equation of the spring-mass system is as follows:

$$\ddot{y}(t) + 5\dot{y}(t) + 4y(t) = -2e^{-3t} - 4e^{-5t}, \quad y(0) = 0, \quad \dot{y}(0) = 5 \tag{3}$$

1. Using Laplace-transform, determine $y(t)$, the time function of the position of the mass.
2. Is there any oscillation in the movement of the mass?
3. Determine $\lim\limits_{t\to\infty} y(t)$.
4. Using Laplace transform, determine the transfer function $(H(s))$ of the system if $u(t) := F(t)$ is considered as an input signal and $y(t)$ is the output signal.
5. Using $H(s)$, give a controllable state-space realization of the system.

Compulsory only for the studends of the course “01TG” (csak a tehetséggondozásban résztvevő hallgatók számára kötelező).

Problem 4. (Basic numeric operations in system's theory)
We consider a strictly proper ($D = 0$) 3-dimensional SISO state-space model

$$\begin{aligned}\dot{x} &= Ax + Bu,\\ y &= Cx,\end{aligned} \qquad A = \begin{pmatrix} -2 & -3 & -8 \\ 0 & -3 & -6 \\ 0 & 1 & -3 \end{pmatrix}, \quad B = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}. \tag{4}$$

1. Compute the diagonal matrix D and invertible matrix S such that $A = SDS^{-1}$. Use the function `eig`.
2. Solve the state-space model using the numeric built-in solver `ode45` if
   (a) $x(0) \neq 0$, $u(t) = 0$. Try different values for the initial conditions $x(0)$, and plot the resulting trajectories $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}$ alongside each other using a 3D plot (`plot3`). See Figure 1.
   (b) $x(0) = 0$, $u(t) = \sin(\omega t)$. Try different values for ω.

   Additionally, in both cases, plot the state variables $x_i(t)$ in the function of the time.
3. Plot the impulse response $h(t)$ of the system (`impulse`).

4. Determine the transfer function $H(s)$ of the system from u to y. Give the coefficients of the numerator $b(s)$ and denominator $a(s)$ of the transfer function $H(s)\frac{b(s)}{a(s)}$. You can use functions **ss, tf, tfdata** or **ss2tf** (deprecated).

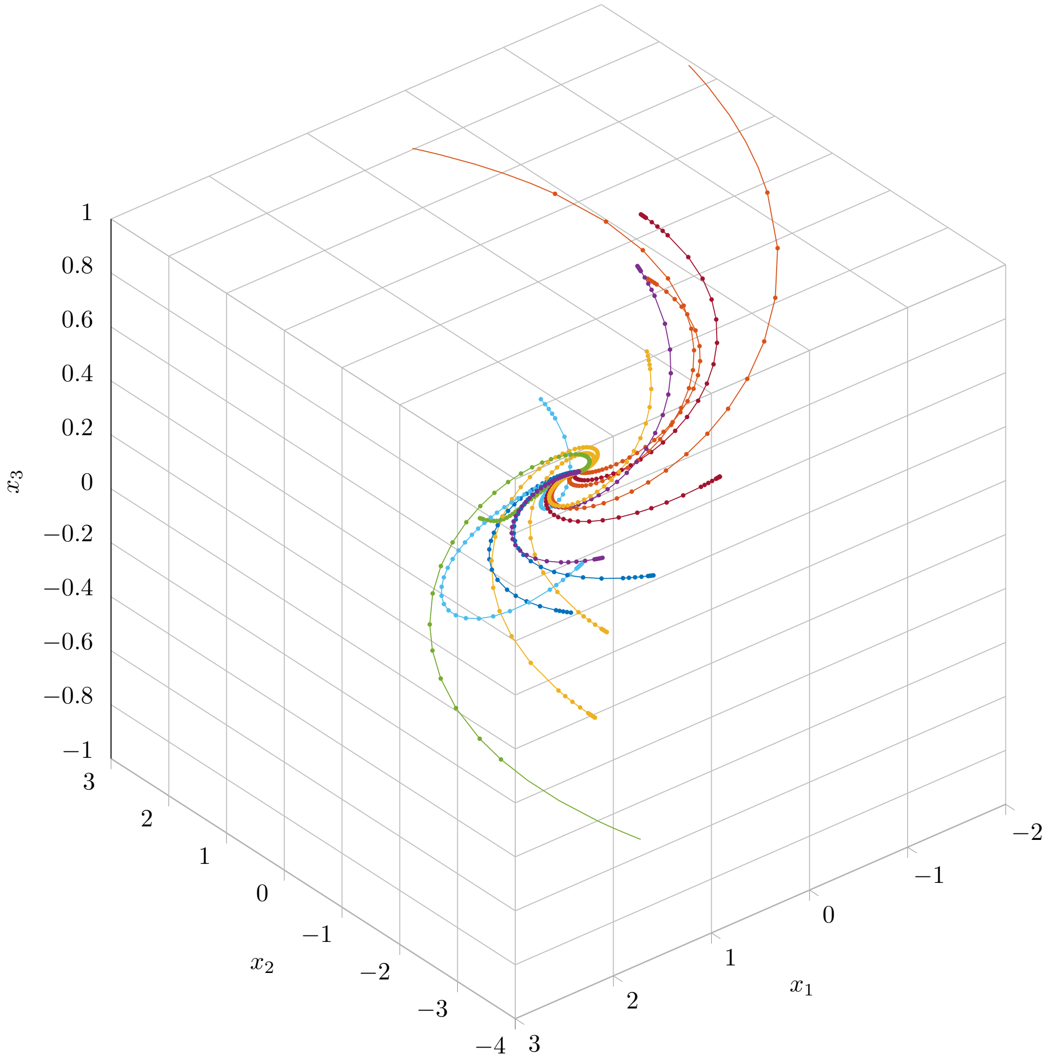


Figure 1. Phase portrait of the 3-dimensional system given in Problem 4.