

# Computer controlled systems

## Lecture 3

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### 1 System representations

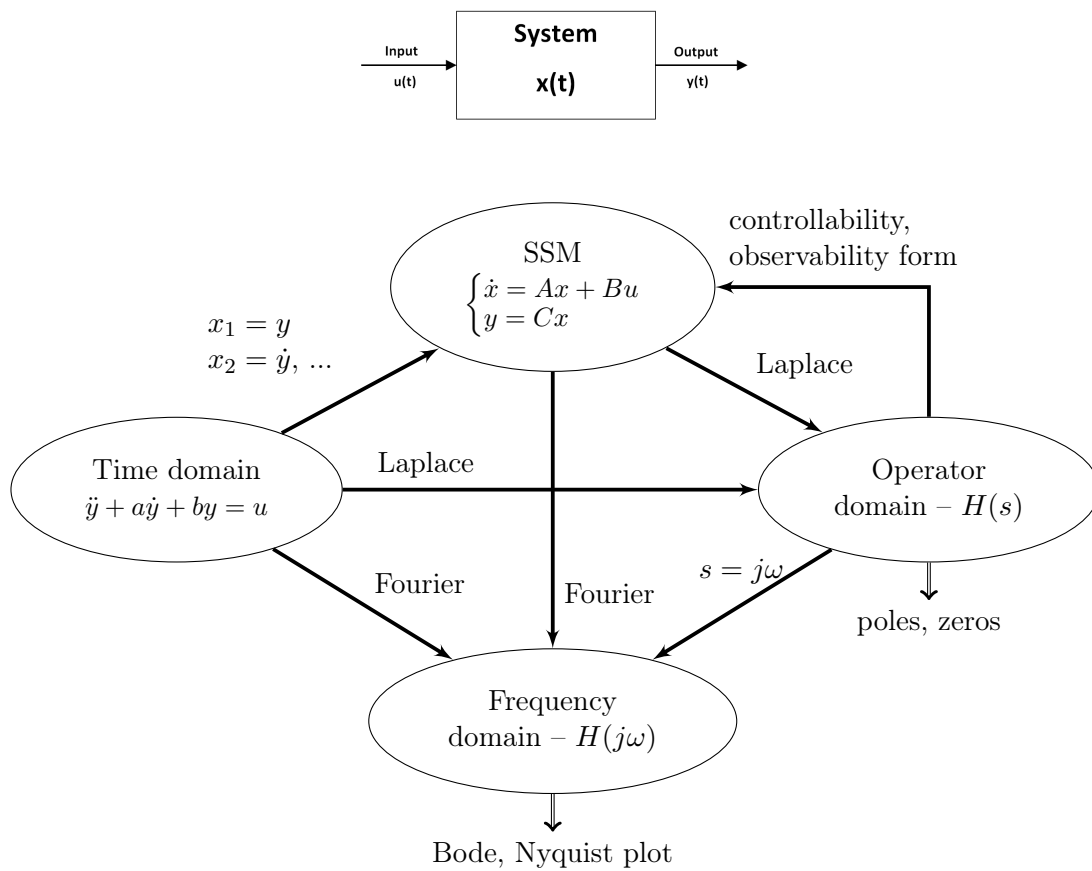


Figure 1. System representations, SSM (State Space Model) – ÁTM (Állapottér model)

1.1 Time domain → Operator domain

**Example 1.** The system's differential equation:

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = 2\ddot{u}(t) - 3\dot{u}(t) + u(t)$$

Assume that the initial conditions are all zero. The Laplace transform of the above equation is:

$$s^2Y(s) + 3sY(s) + 2Y(s) = 2s^2U(s) - 3sU(s) + U(s)$$

$$(s^2 + 3s + 2)Y(s) = (2s^2 - 3s + 1)U(s)$$

Controller, observer form (in advance)

From this, we obtain the system's transfer function:

$$H(s) = \frac{Y(s)}{U(s)} = \frac{2s^2 - 3s + 1}{s^2 + 3s + 2} = \underbrace{\frac{-9s - 3}{s^2 + 3s + 2}}_{C(sI-A)^{-1}B} + \underbrace{2}_D \Rightarrow \begin{cases} \dot{x} = \begin{pmatrix} -3 & -2 \\ 1 & 0 \end{pmatrix}x + \begin{pmatrix} 1 \\ 0 \end{pmatrix}u \\ y = \begin{pmatrix} -9 & -3 \end{pmatrix}x + 2u \end{cases} \text{ ctr. form}$$

$$\begin{cases} \dot{x} = \begin{pmatrix} -3 & 1 \\ -2 & 0 \end{pmatrix}x + \begin{pmatrix} -9 \\ -3 \end{pmatrix}u \\ y = \begin{pmatrix} 1 & 0 \end{pmatrix}x + 2u \end{cases} \text{ obs. form}$$

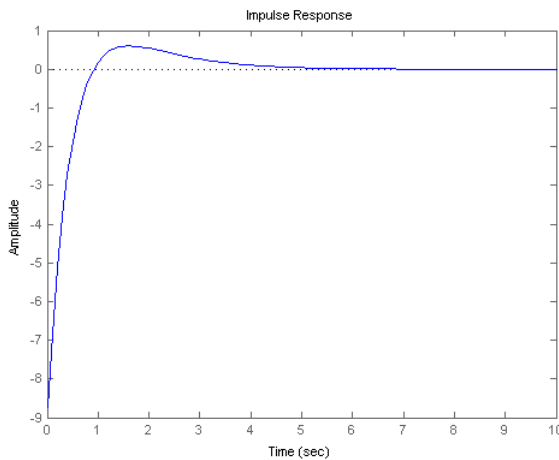
From  $H(s)$ , we can determine the system's impulse response function (using inverse Laplace transformation):

$$H(s) = 2 + \frac{-9s - 3}{(s + 1)(s + 2)} = 2 + \frac{C_1}{s + 1} + \frac{C_2}{s + 2}$$

$$C_i = \lim_{s \rightarrow \alpha_i} (s - \alpha_i)H(s)$$

$$C_1 = 6 \quad C_2 = -15$$

$$h(t) = \mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\left\{2 + \frac{6}{s + 1} + \frac{-15}{s + 2}\right\} = 2\delta(t) + 6e^{-t} - 15e^{-2t}$$



In a strictly proper system the input does not affect the output *directly*. In the operator domain, this means that the degree of the transfer function's numerator is less than the degree of the denominator. In the state space model, matrix  $D$  is zero if the system is strictly proper.

Figure 2. Impulse response  $h(t)$  of the system.

**Matlab 1.** conv,deconv

Polynomial multiplication and division

Polynome multiplication

$$(s^2 + 3s + 2) \cdot (s + 4) = s^3 + 7s^2 + 14s + 8$$

```
>> C = conv([1 3 2],[1 4])
```

```
C = 1    7    14    8
```

Polynome division

$$\frac{2s^2 - 3s + 1}{s^2 + 3s + 2} = \frac{-9s - 3}{s^2 + 3s + 2} + 2$$

```
>> [Q,R] = deconv([2 -3 1],[1 3 2])
```

```
Q = 2
R = 0    -9    -3
```

**Matlab 2.** ss2tf

Compute the transfer function of a SSM

```
>> [num,den] = ss2tf([-3 -2 ; 1 0],[1 0],[-9 -3],2)
```

```
num = 2    -3    1
```

```
den = 1    3    2
```

## 1.2 Time domain $\rightarrow$ SSM

State space model: *system* of first order linear differential equations (elsőrendű differenciálegyenletekből álló egyenletrendszer)

$$\dot{x} = f(x) + g(x)u, \quad \text{where } f : \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad g : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times r}, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^r. \quad (1)$$

Linear case:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \quad (2)$$

**Example 2.** It is given the following second order linear scalar differential equation

$$\ddot{y} = -y$$

We introduce the following notation:

$$x_1 = y, \quad x_2 = \dot{y} = \dot{x}_1$$

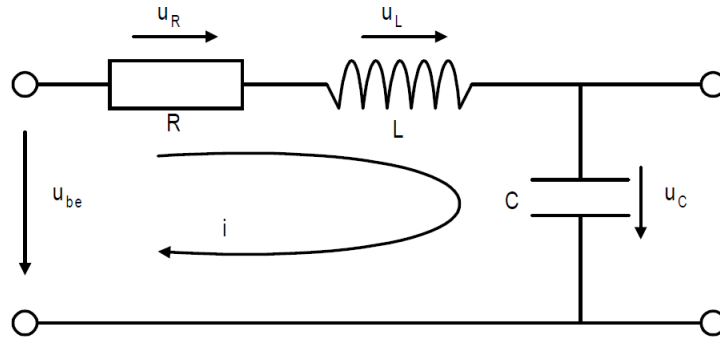
SSM:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \ddot{x}_1 = \ddot{y} = -y = -x_1 \end{aligned}$$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \rightarrow A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad C = (1 \quad 0)$$

Usually, the procedure for (time domain  $\rightarrow$  SSM) is the following:

Linear higher order scalar differential equation with constant coefficient  $\rightarrow$  compute its transfer function  $H(s) \rightarrow$  SSM (Controller or Observer form).

**Example 3.**

The system's differential equation ( $u_{be}$  stands for  $u_{in}$  – is the input voltage):

$$u_{be} = u_R + u_L + u_C$$

$$i = C \cdot \frac{du_C}{dt}, \quad u_L = L \cdot \frac{di}{dt}, \quad u_R = R \cdot i$$

State equations (let the state variables be:  $i, u_C$ ):

$$\begin{cases} u_{be} = R \cdot i + L \cdot \frac{di}{dt} + u_C \\ i = C \cdot \frac{du_C}{dt} \end{cases} \Rightarrow \begin{cases} \frac{di}{dt} = -\frac{R}{L}i - \frac{1}{L}u_C + \frac{1}{L}u_{be} \\ \frac{du_C}{dt} = \frac{1}{C}i \end{cases} \quad (3)$$

In matrix form:

$$\begin{pmatrix} \frac{di}{dt} \\ \frac{du_C}{dt} \end{pmatrix} = \begin{pmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{pmatrix} \cdot \begin{pmatrix} i \\ u_C \end{pmatrix} + \begin{pmatrix} \frac{1}{L} \\ 0 \end{pmatrix} u_{be}$$

Hence, the state vector is  $x(t) = \begin{pmatrix} i \\ u_C \end{pmatrix}$ ,  $u(t) = u_{be}$

We must define the output of the system:  $y(t) = u_C = (0 \ 1) \begin{pmatrix} i \\ u_C \end{pmatrix}$

Now we consider a concrete numerical example. Let  $R = 1.5\Omega$ ,  $L = 0.25H$ ,  $C = 0.5F$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -6 & -4 \\ 2 & 0 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 4 \\ 0 \end{pmatrix} u$$

$$y = (0 \ 1) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$A = \begin{pmatrix} -6 & -4 \\ 2 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \quad C = (0 \ 1)$$

In Figure 3, you can see the impulse and step response of the system. Furthermore, Figure 4 illustrates the system's output in case of a sinusoidal input function  $u(t) = 2 \sin(3t)$  (szinuszos gerjesztés).

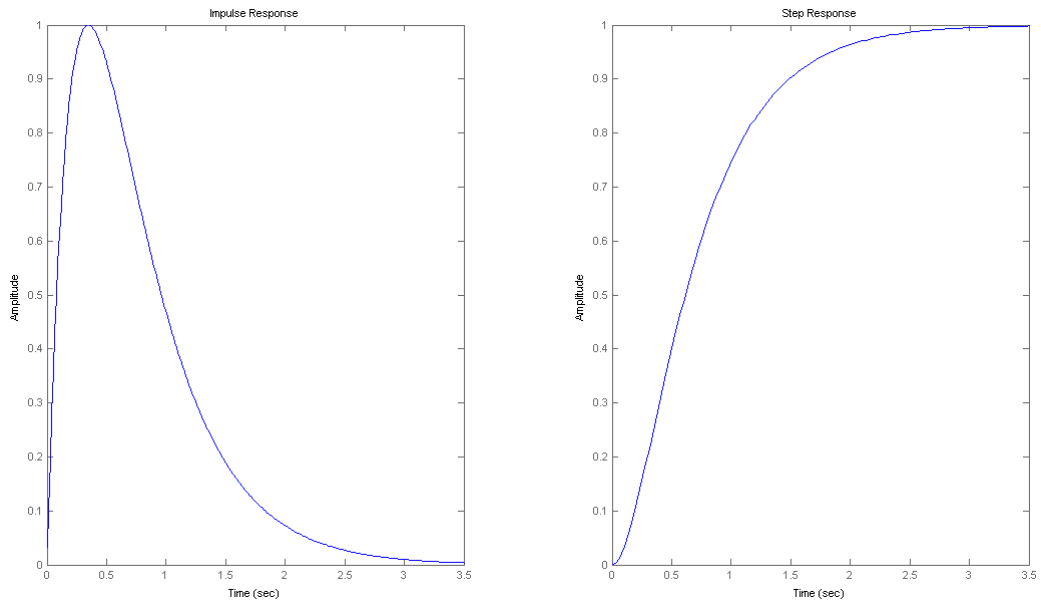


Figure 3. Impulse and step response of the RLC circuit

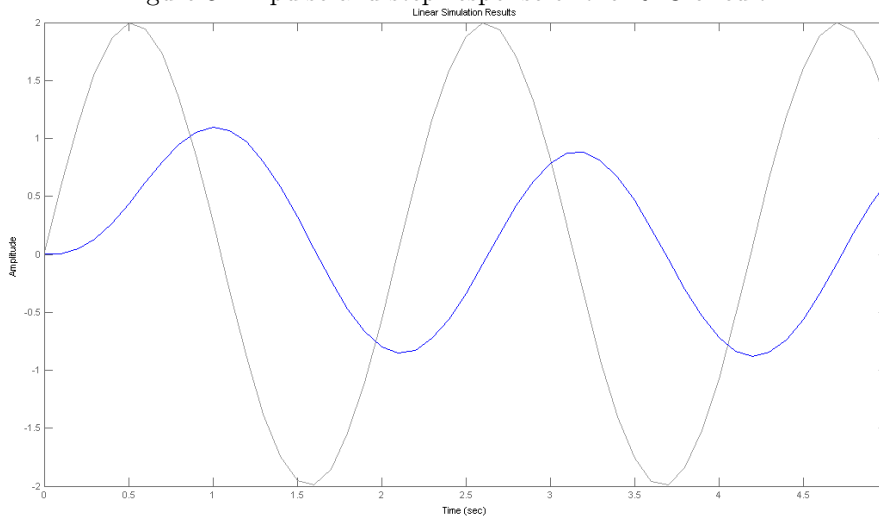


Figure 4. Excitation (gerjesztés) of the RLC circuit with a sinusoidal input function. In case of  $f = 3\text{Hz}$  frequency signal the system's gain is  $g = 0.4438$ , phase shift is  $\phi = -93.1798^\circ$ . HU: 3 Hz esetén a rendszer egyenáramú erősítése  $g = 0.4438$ , fáziseltolása  $\phi = -93.1798^\circ$ .

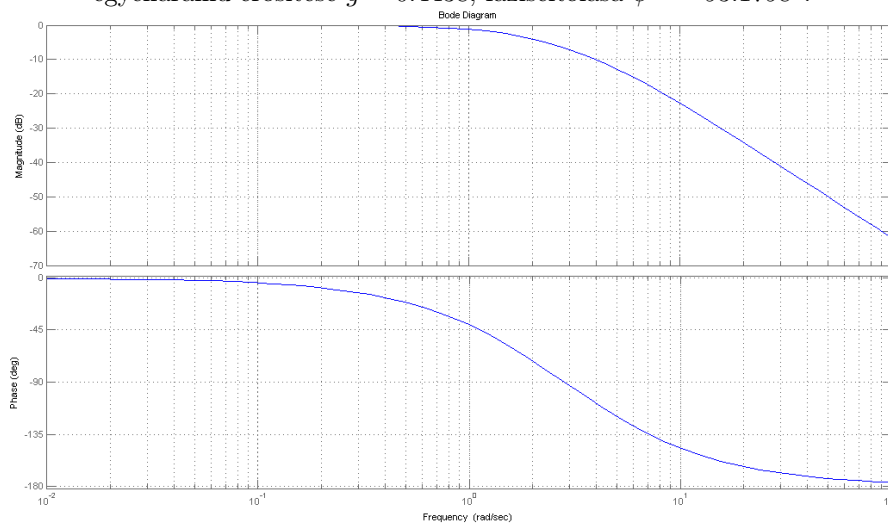


Figure 5. Bode diagram of the system describing the RLC circuit.

**Example 4.** (Diagonal SSM)

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -6 & -4 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 4 \\ 0 \end{pmatrix} u$$

$$y = \begin{pmatrix} 0 & 1 \end{pmatrix} x$$

We compute the eigenvalue/eigenvector decomposition (alternatively: *spectral decomposition*) of matrix A

$$\begin{vmatrix} \lambda + 6 & 4 \\ -2 & \lambda \end{vmatrix} = (\lambda + 6)\lambda + 8 = (\lambda + 2)(\lambda + 4)$$

Eigenvalues:  $\lambda_1 = -2$ ,  $\lambda_2 = -4$

Eigenvectors in case of  $\lambda_1 = -2$ :

$$\begin{pmatrix} -6 & -4 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = -2 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \rightarrow \begin{array}{l} -6v_1 - 4v_2 = -2v_1 \rightarrow v_2 = -v_1 \\ 2v_1 = -2v_2 \end{array}$$

$$v = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot p \quad p \in \mathbb{R} \setminus \{0\}$$

Eigenvectors in case of  $\lambda_2 = -4$ :

$$\begin{pmatrix} -6 & -4 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = -4 \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \rightarrow \begin{array}{l} -6w_1 - 4w_2 = -4w_1 \rightarrow w_1 = -2w_2 \\ 2w_1 = -4w_2 \end{array}$$

$$w = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \cdot q \quad q \in \mathbb{R} \setminus \{0\}$$

Transformation matrix:

$$T = S^{-1} \quad T^{-1} = S$$

$$T^{-1} = (v \ w) = \begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix} \rightarrow T = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix}$$

State space transformation:

$$\bar{A} = TAT^{-1} = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} -6 & -4 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & -4 \end{pmatrix}$$

$$\bar{B} = TB = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 4 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ -4 \end{pmatrix}$$

$$\bar{C} = CT^{-1} = \begin{pmatrix} -1 & 1 \end{pmatrix}$$

1.3 SSM  $\rightarrow$  Operator domain

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \rightarrow sIX(s) = AX(s) + BU(s) \rightarrow X(s) = (sI - A)^{-1}BU(s) \\ x(0) &= 0 \quad Y(s) = CX(s) + DU(s) = (C(sI - A)^{-1}B + D)U(s) \end{aligned}$$

$$H(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D$$

**Example 5.**

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -6 & -4 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 4 \\ 0 \end{pmatrix} u$$

$$y = (0 \ 1) x$$

$$\begin{aligned} H(s) &= (0 \ 1) \begin{pmatrix} s+6 & 4 \\ -2 & s \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ 0 \end{pmatrix} = \\ &= (0 \ 1) \frac{1}{(s+6)s+8} \begin{pmatrix} s & -4 \\ 2 & s+6 \end{pmatrix} \begin{pmatrix} 4 \\ 0 \end{pmatrix} = \frac{1}{s^2+6s+8} (0 \ 1) \begin{pmatrix} 4s \\ 8 \end{pmatrix} = \frac{8}{(s+2)(s+4)} \end{aligned}$$

**Proposition 1.** (special case – if the SSM is diagonal, i.e. matrix  $A$  is diagonal)

$$\begin{aligned} H(s) &= C(sI - A)^{-1}B = (c_1 \cdots c_n) \begin{pmatrix} s - \lambda_1 & 0 & \cdots & 0 \\ 0 & s - \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & s - \lambda_n \end{pmatrix}^{-1} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \\ &= (c_1 \cdots c_n) \begin{pmatrix} \frac{1}{s-\lambda_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{s-\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{s-\lambda_n} \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \frac{c_1 b_1}{s - \lambda_1} + \frac{c_2 b_2}{s - \lambda_2} + \cdots + \frac{c_n b_n}{s - \lambda_n} = \sum_{i=1}^n \frac{c_i b_i}{s - \lambda_i} \end{aligned}$$

**Example 6.**

$$\dot{x} = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -6 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \\ -10 \\ 2 \end{pmatrix} u$$

$$y = (6 \ 8 \ 2 \ -1) x$$

$$H(s) = \frac{0 \cdot 6}{s-4} + \frac{1 \cdot 8}{s+3} + \frac{(-10) \cdot 2}{s+2} + \frac{2 \cdot (-1)}{s+6} = \frac{8}{s+3} - \frac{20}{s+2} - \frac{2}{s+6}$$

1.4 Operator domain  $\rightarrow$  SSM : Controller form

**Theorem 2.** (Controller, observer form – only SISO)

$$H(s) = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \dots + b_0}{1s^n + a_{n-1}s^{n-1} + \dots + a_0} + D = \frac{b(s)}{a(s)} + D, \quad D \in \mathbb{R}$$

The denominator must always be monic (with leading coefficient 1).

A nevezőben szereplő polinom vezéregyütthatója mindig 1 kell, hogy legyen!

$$\text{controller form: } \begin{cases} \dot{x} = \begin{pmatrix} -a_{n-1} & -a_{n-2} & \dots & -a_1 & -a_0 \\ 1 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} u \\ y = (b_{n-1} \ b_{n-2} \ \dots \ b_0) x + Du \end{cases} \quad (4)$$

$$\text{observer form: } \begin{cases} \dot{x} = \begin{pmatrix} -a_{n-1} & 1 & \dots & 0 & 0 \\ -a_{n-2} & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ -a_1 & 0 & \dots & 1 & 0 \\ -a_0 & 0 & \dots & 0 & 0 \end{pmatrix} x + \begin{pmatrix} b_{n-1} \\ b_{n-2} \\ \vdots \\ b_0 \end{pmatrix} u \\ y = (1 \ 0 \ \dots \ 0) x + Du \end{cases} \quad (5)$$

**Example 7.**

$$H(s) = \frac{4s + 38}{(s+1)(s+2)(2s+6)} = \frac{4s + 38}{2s^3 + 12s^2 + 22s + 12} = \frac{2s + 19}{s^3 + 6s^2 + 11s + 6}$$

$$\dot{x} = \begin{pmatrix} -6 & -11 & -6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} u$$

$$y = (0 \ 2 \ 19) x$$

*Remark.* The controller form produces a controllable SSM!

A controller form mindig irányítható ÁTM-et eredményez! (Lásd később)

Check whether it leads, indeed, to the initial transfer function:

$$\begin{aligned} H(s) &= C(sI - A)^{-1}B = (0 \ 2 \ 19) \begin{pmatrix} s+6 & 11 & 6 \\ -1 & s & 0 \\ 0 & -1 & s \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \\ &= (0 \ 2 \ 19) \frac{1}{s^2(s+6) + 6 + 11s} \begin{pmatrix} s^2 & -11s - 6 & 6s \\ s & s(s+6) & -6 \\ 1 & s+6 & s(s+6) + 11 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \\ &= \frac{1}{s^3 + 17s + 6} (0 \ 2 \ 19) \begin{pmatrix} s^2 \\ s \\ 1 \end{pmatrix} = \frac{2s + 19}{s^3 + 6s^2 + 11s + 6} \end{aligned}$$

**Only SISO case.** The controller and observer forms are very similar:

$$A_c = A_o^T, \quad B_c = C_o^T, \quad C_c = B_o^T \quad (6)$$

SISO  $\Rightarrow C(sI - A)^{-1}B \in \mathbb{R}$ , thus

$$C(sI - A)^{-1}B = (C(sI - A)^{-1}B)^T = B^T(sI - A)^{-T}C^T = B^T(sI - A^T)^{-1}C^T \quad (7)$$