Computer controlled systems

Lecture 2

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Types

- Convolution of two functions
- $\ddot{y} + a\dot{y} + by = e^{-at},\ y(0),\ \dot{y}(0)$ type initial value tasks (solution by Laplace transform)
- partial fraction decomposition
- computation of the transfer function time-constant form for systems given in the form $\ddot{y}+a\dot{y}+by = u(t)$
- solution of initial values problems of type $\dot{x} = Ax,\ x(0)$ using Laplace transform
- scompute the transfer function ($H(s)$) for a system given as state space model ($\dot{x} = Ax + Bu,\ y = Cx$)
- solution of the state space model, given both the input and the initial values – impulse response ($h(t)$), response to the unit step function

1. Laplace transform

Definition: $f(t) \to F(s) \quad s \in \mathbb{C}$

$$F(s) = \mathfrak{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\mathrm{d}t \tag{1}$$

Based on the properties of the integral the laplace transform is a linear mapping.

1.1. Rules

1. Convolution in time domain: $\mathfrak{L}\{(f * g)(t)\} = F(s)G(s),$

   ahol $F(s) = \mathfrak{L}\{f(t)\}$, $G(s) = \mathfrak{L}\{g(t)\}$, $(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\mathrm{d}\tau$. Derivation:

$$\begin{aligned}\mathfrak{L}\{(f * g)(t)\} &= \int_0^\infty \int_0^t f(\tau)g(t-\tau)\mathrm{d}\tau\ e^{-st}\mathrm{d}t = \int_0^\infty \int_0^\infty f(\tau)g(t-\tau)e^{-st}\mathrm{d}t\ \mathrm{d}\tau \\ &= \int_0^\infty \int_0^\infty g(t-\tau)e^{-s(t-\tau)}\mathrm{d}t\ f(\tau)e^{-s\tau}\mathrm{d}\tau = \int_0^\infty \int_0^\infty g(\vartheta)e^{-s\vartheta}\mathrm{d}\vartheta\ f(\tau)e^{-s\tau}\mathrm{d}\tau \\ &= \int_0^\infty g(\vartheta)e^{-s\vartheta}\mathrm{d}\vartheta \int_0^\infty f(\tau)e^{-s\tau}\mathrm{d}\tau = \end{aligned} \tag{2}$$

We will deal with functions for which $f(t) = g(t) = 0$ for all $t < 0$, hence

$$\int_0^t f(\tau)g(t-\tau)\mathrm{d}\tau = \int_0^\infty f(\tau)g(t-\tau)\mathrm{d}\tau \quad \text{mivel } g(t-\tau) = 0 \text{ bármely } \tau > t \tag{3}$$

It was also used during the above derivation (change of variables: $\vartheta = t - \tau$):

$$\int_0^\infty g(t-\tau)e^{-s(t-\tau)}\mathrm{d}t = \int_{-\tau}^\infty g(\vartheta)e^{-s\vartheta}\mathrm{d}\vartheta = \int_0^\infty g(\vartheta)e^{-s\vartheta}\mathrm{d}\vartheta \quad \text{mivel } g(t<0) = 0 \tag{4}$$

2. Time derivative:

   $\boxed{\mathfrak{L}\{\dot{y}(t)\} = sY(s) - y(0),}$ ahol $Y(s) = \mathfrak{L}\{y(t)\}$. Derivation:

$$\int_0^\infty \dot{y}(t)e^{-st}\mathrm{d}t = y(t)e^{-st}\Big|_0^\infty - (-s)\int_0^\infty y(t)e^{-st}\mathrm{d}t = -y(0) + s\mathfrak{L}\{y(t)\} \tag{5}$$

3. Second derivative according to the time variable:

   $\boxed{\mathfrak{L}\{\ddot{y}(t)\} = s^2Y(s) - \dot{y}(0) - sy(0).}$ Derivation:

$$\mathfrak{L}\{\ddot{y}(t)\} = s\mathfrak{L}\{\dot{y}(t)\} - \dot{y}(0) = s^2Y(s) - sy(0) - \dot{y}(0) \tag{6}$$

1.2. limit theorems

1. $\boxed{y(0) = \lim_{s\to\infty} sY(s))}$ (Initial value theorem)

   Proof. Let us take the limit of both the left and right sides of the rule of derivation $s \to \infty$:

$$\lim_{s\to\infty}\int_0^\infty \underbrace{e^{-st}}_{\to 0}\dot{y}(t)\mathrm{d}t = \lim_{s\to\infty}(sY(s) - y(0)) \quad \Rightarrow \quad y(0) = \lim_{s\to\infty} sY(s) \tag{7}$$

2. $\boxed{y(\infty) = \lim_{s\to 0} sY(s)}$

   Proof. Let us take the derivation rule

$$\int_0^\infty \dot{y}(t)e^{-st}\mathrm{d}t = sY(s) - y(0) \tag{8}$$

   and consider the limit of both sides $s \to 0$:

$$\lim_{s\to 0}\int_0^\infty \underbrace{e^{-st}}_{\to 1}\underbrace{\dot{y}(t)\mathrm{d}t}_{\mathrm{d}y(t)} = \lim_{s\to 0}(sY(s) - y(0)) \tag{9}$$

$$\lim_{s\to 0}\int_0^\infty \mathrm{d}y(t) = \lim_{s\to 0} sY(s) - y(0) \tag{10}$$

$$y(\infty) - y(0) = \lim_{s\to 0} sY(s) - y(0) \quad \Rightarrow \quad y(\infty) = \lim_{s\to 0} sY(s) \tag{11}$$

1.3. Laplace transform for significant functions

1. $\boxed{\mathfrak{L}\{\delta(t)\} = 1}$ derivation: $\int_0^\infty \delta(t)e^{-st}\mathrm{d}t = \lim_{T\to 0}\frac{1}{T}\int_0^T e^{-st}\mathrm{d}t = \frac{1}{s}\underbrace{\lim_{T\to 0}\frac{1-e^{-sT}}{T}}_{\text{L'Hospital: } s} = 1$

2. $\boxed{\mathfrak{L}\{1(t)\} = \frac{1}{s}}$ derivation: $\int_0^\infty 1(t)e^{-st}\mathrm{d}t = \left[\frac{e^{-st}}{-s}\right]_0^\infty = 0 - \left(-\frac{1}{s}\right) = \frac{1}{s}$,

   where $1(t)$ unit step function, for this the $u(t)$ notation is laso commonly used, but in this case $u(t)$ denotes the input of the system.

3. $\mathfrak{L}\{t \cdot 1(t)\} = \frac{1}{s^2}$ (unit step velocity function)

4. $\boxed{\mathfrak{L}\{e^{-at}\} = \frac{1}{s+a},}$ it is the most useful for inverse Laplace transform.
5. $\mathfrak{L}\{e^{-t/T}\} = \frac{1}{s+1/T} = \frac{T}{1+sT}$, it is another form of the previous case.
   Levezetés: $\int_0^\infty e^{-t/T} e^{-st} \mathrm{d}t = \int_0^\infty e^{-(s+1/T)t} \mathrm{d}t \left[\frac{e^{-(s+1/T)t}}{-(s+1/T)}\right]_0^\infty = \frac{1}{s+1/T} = \frac{T}{1+sT}$

   pole-zero from $\frac{1}{s+1/T}$

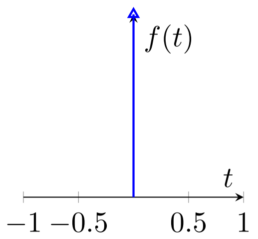
   time-constant form $\frac{T}{1+sT}$
6. $\mathfrak{L}\{1 - e^{-t/T}\} = \frac{1}{s(1+sT)}$ (time-constant form)
7. $\mathfrak{L}\left\{\frac{1}{T_1-T_2}(e^{-t/T_1} - e^{-t/T_2})\right\} = \frac{1}{(1+sT_1)(1+sT_2)}$
8. $\mathfrak{L}\{e^{at}\sin(bt)\} = \dfrac{b}{(s-a)^2+b^2}$
9. $\mathfrak{L}\{e^{at}\cos(bt)\} = \dfrac{s-a}{(s-a)^2+b^2}$

1.4. Inverse Laplace transform

$f(t) = \mathfrak{L}^{-1}\{F(s)\} = \frac{1}{2\pi j} \lim_{T\to\infty} \int_{c-jT}^{c+jT} F(s) e^{ts} ds$
where $c \in \mathbb{R}$ is greater than the real parts of F(s)'s singularities.

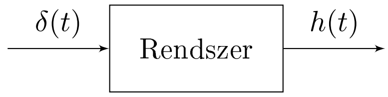
1.5. Input, system response

1. **Dirac impulse**



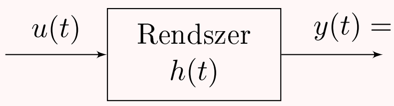
$$f_\tau(t) = \begin{cases} \frac{1}{\tau} & \text{ha } 0 \leq t < \tau \\ 0 & \text{egyébként} \end{cases} \qquad \delta(t) = \lim_{\tau \to 0+} f_\tau(t)$$

2. **the output (response) of the system to the Dirac impulse (impulse response)**: $h(t)$
   E.g.. if I strike on a trapdoor ($\delta(t)$) , then it will dampedly oscillate($h(t)$).



   Convolutional time-invariance: $\delta(t-\tau)$, $h(t-\tau)$.

3. **The system response to $u(t)$ (transfer function)**: Causal convolution



$$y(t) = (u * h)(t) = \int_0^t h(t-\tau)u(\tau)d\tau$$

Example 1.

Let us compute the convolution of $f(t) = t$ and $g(t) = t^2$:

$$(f * g)(t) = \int_0^t (t-\tau)\tau^2 d\tau = \int_0^t t\tau^2 - \tau^3 d\tau = \left[\frac{t\tau^3}{3} - \frac{\tau^4}{4}\right]_0^t = \frac{t^4}{12} \qquad (12)$$

2. applying laplace transform to solve initial value problems

Example 2. Constant coefficient second order linear differential equation

Solve the following initial value problem:

$$\ddot{y} - 2\dot{y} + 5y = -8e^{-t} \quad y(0) = 2 \quad \dot{y}(0) = 12$$

One can compute the Laplace transform as follows (elementwise).

$$\mathfrak{L}\{\ddot{y}\} - 2\mathfrak{L}\{\dot{y}\} + 5\mathfrak{L}\{y\} = -\frac{8}{s+1} \tag{13}$$

Laplace transform in the case of derivated function: $\mathfrak{L}\{\dot{y}\} = sY(s) - y(0) = sY(s) - 2$. and the second derivative: $\mathfrak{L}\{\ddot{y}\} = s^2Y(s) - sy(0) - \dot{y}(0) = s^2Y(s) - 2s - 12$. such a way the equation (13) has the following form:

$$(s^2Y(s) - 2s - 12) - 2(sY(s) - 2) + 5Y(s) = -\frac{8}{s+1} \tag{14}$$

expressing $Y(s)$ we get:

$$Y(s) = \frac{2s^2 + 10s}{(s^2 - 2s + 5)(s+1)} \xrightarrow{\mathfrak{L}^{-1}} y(t) =? \tag{15}$$

Example 3. Partial fraction decomposition

Tel us solve the following initial value problem:

$$\dddot{y} + 7\ddot{y} + 14\dot{y} + 8y = 0 \quad y(0) = 0 \quad \dot{y}(0) = 0 \quad \ddot{y}(0) = 2$$

The physical interpretation of initial value: nyugalomban lévő testre hat egy gyorsulásvektor (pl. gravitációs gyorsulás). Az előző feladathoz hasonlóan ha vesszük az egyenlet mindkét oldalának The Laplace transform is the following:

$$Y(s) = \frac{2}{(s+1)(s+2)(s+4)} = \frac{C_1}{s+1} + \frac{C_2}{s+2} + \frac{C_3}{s+4} \quad \xrightarrow{\mathfrak{L}^{-1}} \quad y(t) = C_1e^{-t} + C_2e^{-2t} + C_3e^{-4t}$$

I fall the roots of the denominator have single multipülicity, then the following formula can be applied:

$$C_i = \lim_{s\to\alpha_i}(s - \alpha_i)Y(s), \quad \text{ahol } \alpha_i \text{ az } \frac{C_i}{s+\alpha_i} \text{ gyöke}$$

$$C_1 = \lim_{s\to-1}(s+1)Y(s) = \frac{2}{(s+2)(s+4)}\Big|_{s=-1} = \frac{2}{3}$$

$$C_2 = \lim_{s\to-2}(s+2)Y(s) = \frac{2}{(s+1)(s+4)}\Big|_{s=-2} = -1$$

$$C_3 = \lim_{s\to-4}(s+4)Y(s) = \frac{2}{(s+1)(s+2)}\Big|_{s=-4} = \frac{1}{3}$$

$$Y(s) = \frac{\frac{2}{3}}{s+1} + \frac{-1}{s+2} + \frac{\frac{1}{3}}{s+4} \tag{16}$$

Tehát a megoldás:

$$y(t) = \frac{2}{3}e^{-t} - e^{-2t} + \frac{1}{3}e^{-4t} \tag{17}$$

Matlab 1. Inverse Laplace transform `partfrac,ilaplace,residue,poly2sym,sym2poly`

2. Continuation of the example:

$$Y(s) = \frac{2s^2 + 10s}{(s^2 - 2s + 5)(s+1)} = \frac{3s+5}{s^2 - 2s + 5} - \frac{1}{s+1} \quad \Rightarrow \quad y(t) = 3\mathrm{e}^t\left(\cos(2t) + \frac{4\sin(2t)}{3}\right) - \mathrm{e}^{-t}$$

By means of the symbolic toolbox:

```
>> syms s
>> Y = partfrac( (2*s^2 + 10*s) / ((s+1) * (s^2 - 2*s + 5)) )
Y =
(3*s + 5)/(s^2 - 2*s + 5) - 1/(s + 1)
```

```
>> ilaplace(Y)
ans =
3*exp(t)*(cos(2*t) + (4*sin(2*t))/3) - exp(-t)
```

by means of numerical computations:

```
>> Y = expand((s+1) * (s^2 - 2*s + 5))
Y =
s^3 - s^2 + 3*s + 5
>> B = [2 10 0];
>> A = sym2poly(Y)
A =
     1    -1     3     5
>> [r,p,k] = residue(B,A)
r =
   1.5 - 2i
   1.5 + 2i
  -1   + 0i
p =
   1   + 2i
   1   - 2i
  -1   + 0i
k =
     []
```

$$Y(s) = \frac{B(s)}{A(s)} = \sum_i \frac{r_i}{s - p_i} + K(s) = -\frac{1}{s+1} + \frac{1.5 - 2j}{s - 1 - 2j} + \frac{1.5 + 2j}{s - 1 + 2j} \tag{18}$$

```
>> Y = sum(r ./ (s - p)) + poly2sym(k)
Y =
- 1/(s + 1) + (3/2 - 2i)/(s - 1 - 2i) + (3/2 + 2i)/(s - 1 + 2i)
>> latex(Y)
ans =
 - \frac{1}{s + 1} + \frac{\frac{3}{2} - 2\, \mathrm{i}}{s - 1 - 2\, \mathrm{i}} + \frac{\frac{3}{2} + 2\, [...]
>> ilaplace(Y)
ans =
- exp(-t) + exp(t*(1 - 2i))*(3/2 + 2i) + exp(t*(1 + 2i))*(3/2 - 2i)
>> Y = simplify(Y)
ans =
(2*s*(s + 5))/(s^3 - s^2 + 3*s + 5)
>> ilaplace(Y)
ans =
3*exp(t)*(cos(2*t) + (4*sin(2*t))/3) - exp(-t)
```

$$y(t) = -\mathrm{e}^{-t} + \mathrm{e}^{t\,(1-2j)} \left(\frac{3}{2} + 2j\right) e^{t\,(1+2j)} \left(\frac{3}{2} - 2j\right) = 3\mathrm{e}^{t}\left(\cos(2t) + \frac{4\sin(2t)}{3}\right) - \mathrm{e}^{-t} \tag{19}$$

Example 4.

Constant coefficient linear differential equation system

$$\begin{aligned}\dot{x}_1 &= 2x_1 + 3x_2 \\ \dot{x}_2 &= 2x_1 + x_2\end{aligned} \quad \rightarrow \quad \dot{x} = Ax \quad A = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} \quad x_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Solution: $x(t) = e^{At}x_0$, $\boxed{e^{At} = Se^{Dt}S^{-1} = \mathfrak{L}^{-1}\{(sI - A)^{-1}\}.}$

from the eigenvalue-eigenvector decomposition of the first equation (previous practice). moreover in 1 dimension:

$$e^{at} = \mathfrak{L}^{-1}\{(s-a)^{-1}\} = \mathfrak{L}^{-1}\left\{\frac{1}{s-a}\right\} \tag{20}$$

Both of the expressions can be used. In this case the second:

$$\det(sI - A) = \begin{vmatrix} s-2 & -3 \\ -2 & s-1 \end{vmatrix} = (s-2)(s-1) - 6 = s^2 - 3s - 4 = (s-4)(s+1) \tag{21}$$

$$(sI - A)^{-1} = \frac{1}{(s-4)(s+1)} \begin{pmatrix} s-1 & 3 \\ 2 & s-2 \end{pmatrix}$$

According to the linearity of the Laplace transform:

$$e^{At}x_0 = \mathfrak{L}^{-1}\left\{ \begin{pmatrix} \dfrac{3}{(s-4)(s+1)} \\ \dfrac{s-2}{(s-4)(s+1)} \end{pmatrix} \right\}$$

Partial fraction decomposition:

$$\frac{3}{(s-4)(s+1)} = \frac{3}{5}\frac{(s+1)-(s-4)}{(s-4)(s+1)} = \frac{0.6}{s-4} - \frac{0.6}{s+1} \tag{22}$$

Using a simpler method:

$$\frac{s-2}{s^2-3s-4} = \frac{C_3}{s+1} + \frac{C_4}{s-4} \rightarrow C_3 = 0.6 \quad C_4 = 0.4 \tag{23}$$

Finally:

$$x(t) = \mathfrak{L}^{-1}\left\{ \begin{pmatrix} \dfrac{-0.6}{s+1} + \dfrac{0.6}{s-4} \\ \dfrac{0.6}{s+1} + \dfrac{0.4}{s-4} \end{pmatrix} \right\} = \begin{pmatrix} -0.6e^{-t} + 0.6e^{4t} \\ 0.6e^{-t} + 0.4e^{4t} \end{pmatrix} \tag{24}$$

Applying the second formula: $e^{At} = Se^{Dt}S^{-1}$, the decomposition is not required, but the eigenvalues and eigenvectors are necessray.

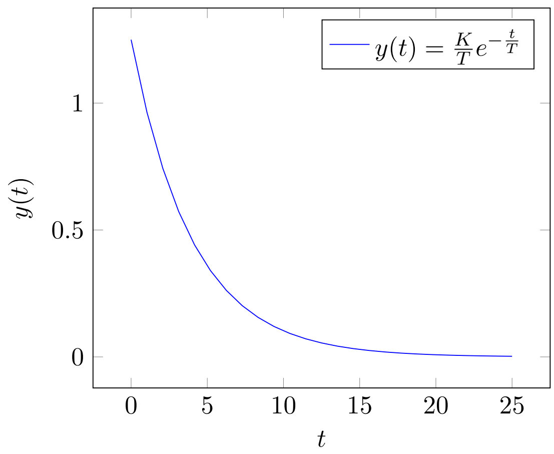
Matlab 2. $\dot{x} = Ax,\ x(0) = x_0$ solution with symbolic toolbox eig,syms,expand,pretty,diag

$$\dot{x} = Ax\ ,\quad x(0) = x_0 \text{ megoldása } x(t) = e^{At}x_0\ ,\quad e^{At} = Se^{Dt}S^{-1} \text{ képlettel} \tag{25}$$

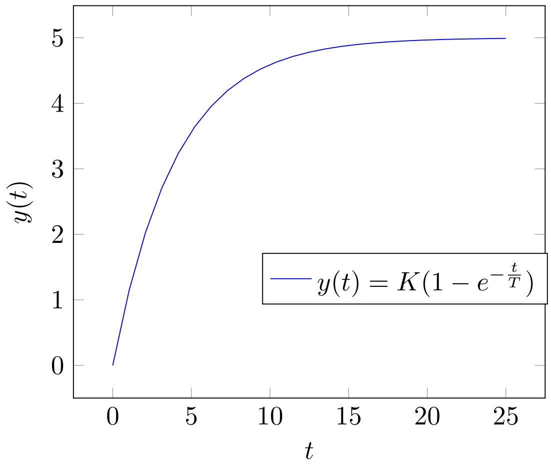
```
syms t real

A = [2 3 ; 2 1];
x0 = [0;1];

[S,D] = eig(A);

SDS_A_iszero = S * D / S - A

exp_Dt = diag(exp(diag(D)*t));
fprintf('\nexp(Dt) = \n\n')
pretty(exp_Dt)

exp_At = expand(S * exp_Dt / S);
fprintf('\n[Matlabbal szamolt sajatvektorok] \nexp(At) = \n\n'), pretty(exp_At)

xt = exp_At * x0;
fprintf('\nA differencialegyenlet megoldasa: x(t) = \n\n')
pretty(expand(xt))
```

Eredmény:

```
exp(Dt) =

/ exp(4 t),    0    \
|                   |
\     0,    exp(-t) /


[Eigenvectors computed by Matlab]
exp(At) =

/ 2 exp(-t)   exp(4 t) 3  exp(4 t) 3   3 exp(-t) \
| --------- + ----------, ---------- - --------- |
|     5           5           5            5     |
|                                                |
| exp(4 t) 2   2 exp(-t)  3 exp(-t)   exp(4 t) 2 |
| ---------- - ---------, --------- + ---------- |
\      5           5          5            5     /

A differencialegyenlet megoldasa: x(t) =

/ exp(4 t) 3   3 exp(-t) \
| ---------- - --------- |
|      5           5     |
|                        |
| 3 exp(-t)   exp(4 t) 2 |
| --------- + ---------- |
\     5           5      /
```



(a) Response to the Dirac impulse



(b) Response to the unit step impulse

Example 5. Applying Laplace transform

The differential equation system describing the system: $T\dot{y} + y = Ku(t) \quad y(0) = 0$
Let us determine the system's response in the followingh cases:

1. $u(t) = \delta(t)$
2. $u(t) = 1(t)$

The Laplace transform of the system: $TsY(s) + Y(s) = KU(s)$, where $T \in \mathbb{R}$ and $K \in \mathbb{R}$ parameters depending on the system. Impulse response function:

$$H(s) = \frac{Y(s)}{U(s)} = \frac{K}{1 + Ts} \tag{26}$$

System's response: $Y(s) = \frac{K}{1+Ts}U(s)$

1. impulse response

$$u(t) = \delta(t) \xrightarrow{\mathfrak{L}} U(s) = 1$$

$$Y(s) = K\frac{1}{1 + Ts} \quad \xrightarrow{\mathfrak{L}^{-1}} \quad y(t) = \mathfrak{L}^{-1}\{Y(s)\} = \frac{K}{T}\mathfrak{L}^{-1}\left\{\frac{1}{s + \frac{1}{T}}\right\} = \frac{K}{T}e^{-t/T}$$

2. transfer function (response to the unit step function)

$$u(t) = 1(t) \xrightarrow{\mathfrak{L}} U(s) = \frac{1}{s}$$

$$Y(s) = K\frac{1}{s(1 + Ts)} \quad \xrightarrow{\mathfrak{L}^{-1}}$$

$$y(t) = \mathfrak{L}^{-1}\{Y(s)\} = \frac{K}{T} \cdot \mathfrak{L}^{-1}\left\{\frac{1}{s} \cdot \frac{1}{s + \frac{1}{T}}\right\} = \frac{K}{T} \cdot (1(t) * e^{\frac{-t}{T}}) = K(1 - e^{-t/T})$$

for the values $K = 5$, $T = 4$ the solution is depicted in the above picture.

3. Állapotegyenlet megoldása

- only excitation ($x_0 = 0, u(t) \neq 0$) $\rightarrow$: Laplace transform
- only initial value ($x_0 \neq 0, u(t) = 0$) $\rightarrow e^{At}x_0$, state trajectories
- both excitation and initial values

Example 6. SSM solution – unit step input

$$A = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad C = \begin{pmatrix} 0 & 1 \end{pmatrix} \quad x_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad u(t) = 1(t)$$

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{27}$$

Applying Laplace transfromation:

$$sX(s) = AX(s) + BU(s) \rightarrow sX(s) - AX(s) = BU(s)$$

$$(sI - A)X(s) = BU(s) \rightarrow X(s) = (sI - A)^{-1}BU(s)$$

$$Y(s) = C(sI - A)^{-1}BU(s)$$

$$H(s) = Y(s)/U(s) = C(sI - A)^{-1}B = \frac{s}{s^2 - 3s - 4} = \frac{s}{(s+1)(s-4)}$$

$$Y(s) = H(s)U(s) = \frac{s}{(s+1)(s-4)} \cdot \frac{1}{s} = \frac{1}{(s+1)(s-4)} = \frac{0.2}{s-4} - \frac{0.2}{s+1}$$

$$y(t) = 0.2e^{4t} - 0.2e^{-t}$$

Example 7. SSM solution – autonomous system

$$A = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad C = \begin{pmatrix} 0 & 1 \end{pmatrix} \quad x_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad u(t) = 0$$

Applying Laplace transfromation:

$$sX(s) - x_0 = AX(s) \quad \rightarrow \quad X(s) = (sI - A)^{-1}x_0 \quad \rightarrow \quad x(t) = \mathfrak{L}^{-1}\{(sI - A)^{-1}\}x_0 = e^{At}x_0$$

$$(sI - A)^{-1} = \frac{1}{(s+1)(s-4)} \cdot \begin{pmatrix} s-1 & 3 \\ 2 & s-2 \end{pmatrix}$$

Output: $y(t) = Cx(t) = C \cdot \mathfrak{L}^{-1}\left\{(sI - A)^{-1}\right\} \cdot x_0 = \mathfrak{L}^{-1}\left\{C(sI - A)^{-1}x_0\right\} = \mathfrak{L}^{-1}\left\{\frac{s-2}{(s+1)(s-4)}\right\} = 0.6e^{-t} + 0.4e^{4t}$

Example 8. SSM solution – unit steo velocity

$$A = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad C = \begin{pmatrix} 0 & 1 \end{pmatrix} \quad x_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad u(t) = t$$

Applying Laplace transformation:

$$sX(s) - x_0 = X(s) + BU(s) \quad \rightarrow \quad X(s) = (sI - A)^{-1}(x_0 + BU(s)) \quad \rightarrow$$

$$x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$$

If $t_0 = 0$, then $e^{A(t-t_0)} = e^{At}$

$$e^{At} = \mathfrak{L}^{-1}\left\{\begin{pmatrix} \frac{s-1}{s^2-3s-4} & \frac{3}{s^2-3s-4} \\ \frac{2}{s^2-3s-4} & \frac{s-2}{s^2-3s-4} \end{pmatrix}\right\} = \mathfrak{L}^{-1}\left\{\begin{pmatrix} \frac{0.6}{s-4} + \frac{0.4}{s+1} & \frac{0.6}{s-4} - \frac{0.6}{s+1} \\ \frac{0.4}{s-4} - \frac{0.4}{s+1} & \frac{0.4}{s-4} + \frac{0.6}{s+1} \end{pmatrix}\right\}$$

$$e^{At} = \begin{pmatrix} 0.6e^{4t} + 0.4e^{-t} & 0.6e^{4t} - 0.6e^{-t} \\ 0.4e^{4t} - 0.4e^{-t} & 0.4e^{4t} + 0.6e^{-t} \end{pmatrix}$$

$$e^{A(t-\tau)} = \begin{pmatrix} 0.6e^{4(t-\tau)} + 0.4e^{-(t-\tau)} & 0.6e^{4(t-\tau)} - 0.6e^{-(t-\tau)} \\ 0.4e^{4(t-\tau)} - 0.4e^{-(t-\tau)} & 0.4e^{4(t-\tau)} + 0.6e^{-(t-\tau)} \end{pmatrix}$$

$$e^{A(t-\tau)}B = \begin{pmatrix} 1.2e^{4(t-\tau)} - 0.2e^{-(t-\tau)} \\ 0.8e^{4(t-\tau)} + 0.2e^{-(t-\tau)} \end{pmatrix} \rightarrow e^{A(t-\tau)}Bu(\tau) = \begin{pmatrix} 1.2e^{4(t-\tau)}\tau - 0.2e^{-(t-\tau)}\tau \\ 0.8e^{4(t-\tau)}\tau + 0.2e^{-(t-\tau)}\tau \end{pmatrix}$$

Elementwise integral:
$\int_0^t c_1 e^{c_2(t-\tau)}\tau d\tau = c_1 e^{c_2 t}\int_0^t e^{-c_2\tau}\tau d\tau = \frac{c_1}{c_2^2}(e^{c_2 t} - c_2 t - 1)$ (Partial integration)

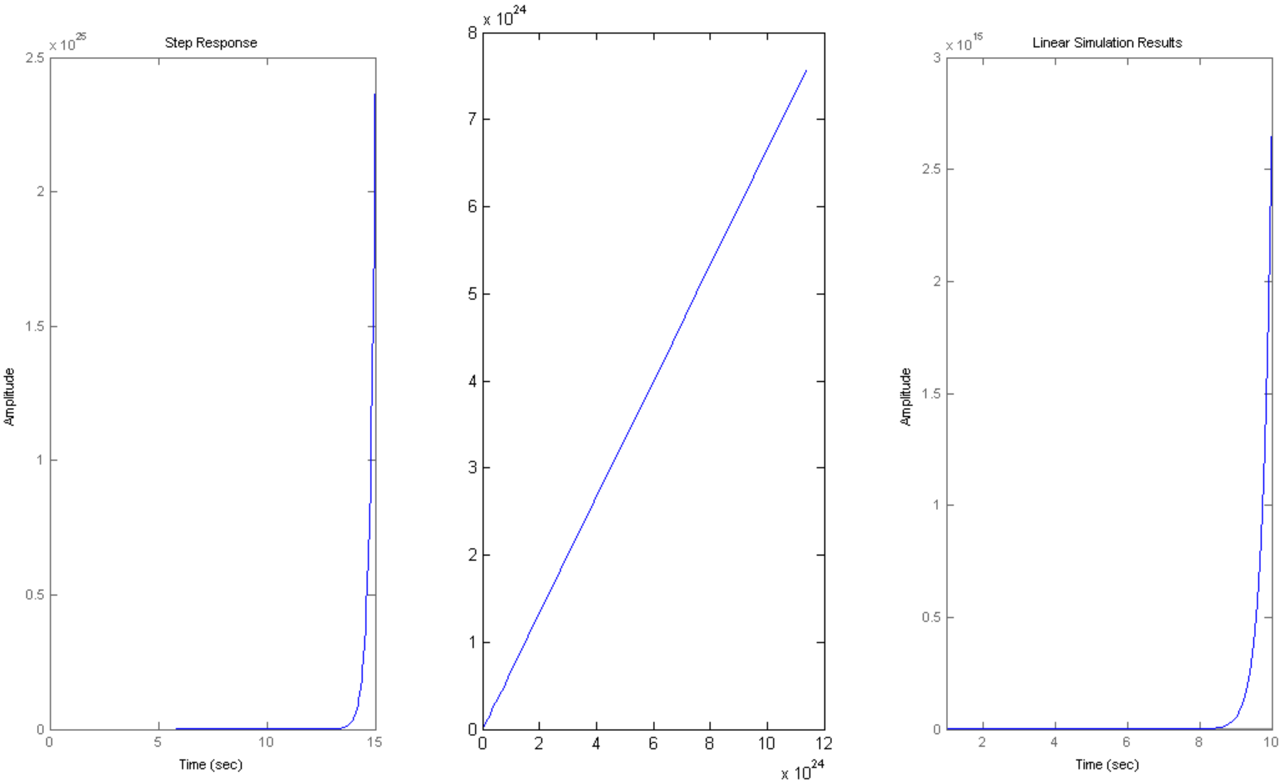
$\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau = \begin{pmatrix} 0.075e^{4t} - 0.2e^{-t} - 0.5t + 0.125 \\ 0.05e^{4t} + 0.2e^{-t} - 0.25 \end{pmatrix}$

$e^{At}x_0$ the same value as in the case of 2. example

$$x(t) = \begin{pmatrix} 0.675e^{4t} - 0.8e^{-t} - 0.5t + 0.125 \\ 0.45e^{4t} + 0.8e^{-t} - 0.25 \end{pmatrix}$$

$$y(t) = Cx(t) = 0.45e^{4t} + 0.8e^{-t} - 0.25$$

A 2. we can see the solution of the three example in order.


Step Response
x 10^25
2.5
2
1.5
1
0.5
0
Amplitude
0
5
10
15
Time (sec)
x 10^24
8
7
6
5
4
3
2
1
0
0
2
4
6
8
10
12
x 10^24
Linear Simulation Results
x 10^15
3
2.5
2
1.5
1
0.5
0
Amplitude
2
4
6
8
10
Time (sec)

2. ábra