

```
> restart;
with(plots):
with(DEtools):
```

Warning, the name changecoords has been redefined

## Intro Lyapunov functions: Simple pendulum

### About this worksheet

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#### Content, Purpose and Use

This worksheet provides procedures to generate pictures that combine solution curves of the DE for a simple (nonlinear) pendulum with the energy function (candidate Lyapunov function).

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The main purpose is to provide powerful images to be used as backdrops for class-discussions.

The 3D-images are best shown live --- wiggle w/ the mouse, but can also be printed for overhead slides.

Most plot ranges, viewing angles have been tuned for the selected initial conditions. Expect substantial rework when changing the DE or IC.

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The implementation uses lots of map and zip commands -- and is NOT meant to be easily read -- instead writing it was a personal MAPLE coding challenge. Particularly nice are the map and zip commands that create several simultaneous plots of solution curves, as defined by numeric-dsolve-output-procedures, and list these to the energy surface and the decay  $dV/dt$ , re-interpreted as a function of the state.

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#### Updates and log of modifications.

none yet.

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### Undamped and damped pendulum: Lyapunov stability

#### The differential equation and phase portrait: undamped case

```
> ude:=diff(theta(t),t)=omega(t),diff(omega(t),t)=-sin(theta(t)):ude;
uic1:=theta(0)=-3/2*Pi,omega(0)=2:uic1;
uic2:=theta(0)=0,omega(0)=2-1/10:uic2;
vars:=theta(t),omega(t):
```

$$\frac{d}{dt}\theta(t) = \omega(t), \quad \frac{d}{dt}\omega(t) = -\sin(\theta(t))$$

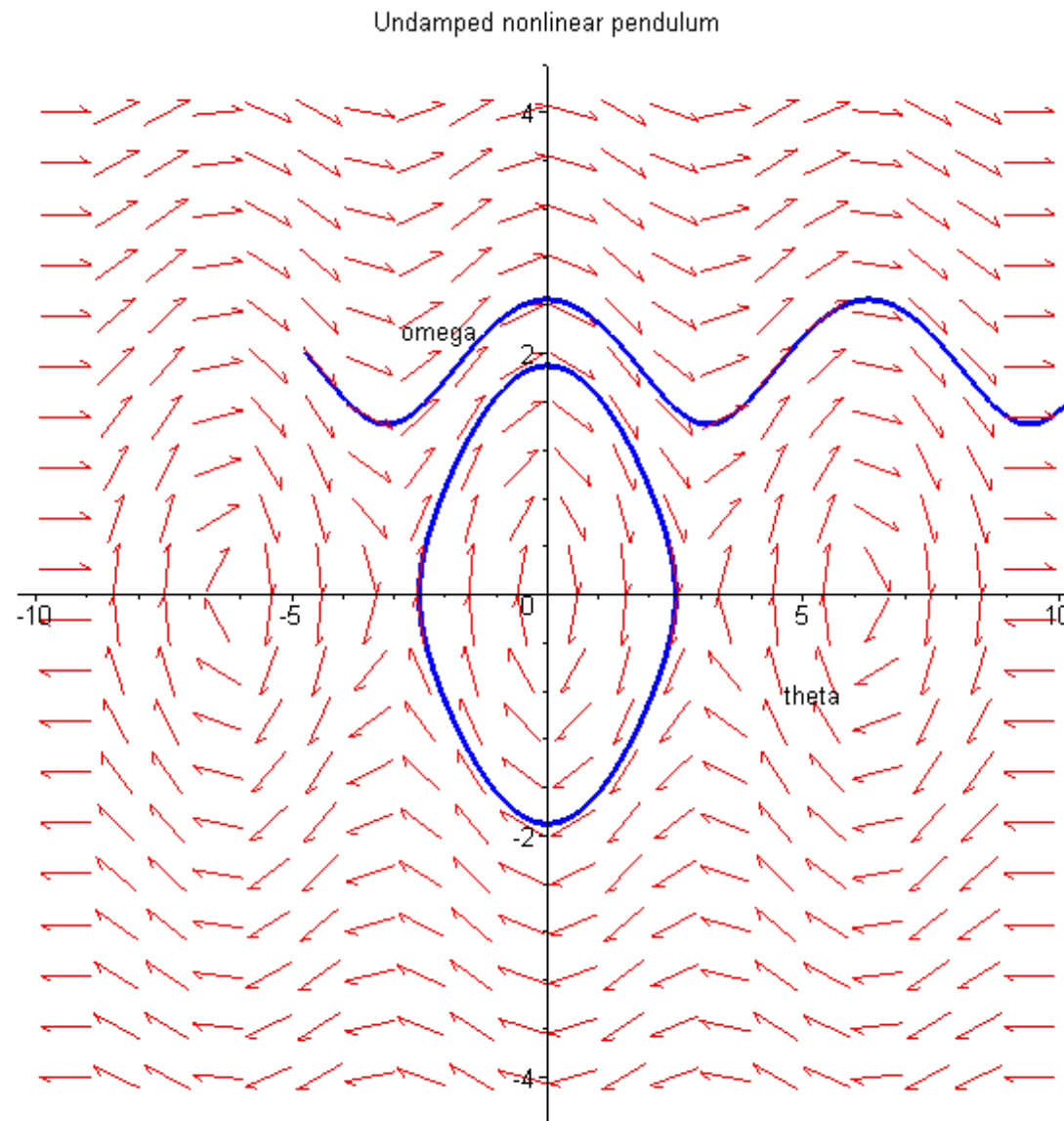
$$\theta(0) = -\frac{3\pi}{2}, \quad \omega(0) = 2$$

$$\theta(0) = 0, \quad \omega(0) = \frac{19}{10}$$

```
> dsolve({ude,uic1},{vars});
```

$$\left\{ \theta(t) = \text{RootOf} \left( \int_{-z}^{-\frac{3\pi}{2}} \frac{1}{\sqrt{2 \cos(\_f) + 4}} d\_f + t \right), \omega(t) = \sqrt{2 \cos \left( \text{RootOf} \left( \int_{-z}^{-\frac{3\pi}{2}} \frac{1}{\sqrt{2 \cos(\_f) + 4}} d\_f + t \right) \right) + 4} \right\}$$

```
> a:=3*Pi;
b:=4;
DEplot([ude],[vars],t=0..20,[[uic1],[uic2]],theta=-a..a,omega=-b..b,
stepsize=.05,method=rkf45,
title=`Undamped nonlinear pendulum`,
color=red,linecolor=blue,arrows=small,thickness=3);
```



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### The differential equation and phase portrait: the damped case

```
> beta:=1/8;
> dde:=diff(theta(t),t)=omega(t),diff(omega(t),t)=-sin(theta(t))-beta*omega(t):dde;
dic1:=theta(0)=-5/2*Pi,omega(0)=3:dic1;
dic2:=theta(0)=0,omega(0)=2-1/10:dic2;
vars:=theta(t),omega(t):
```

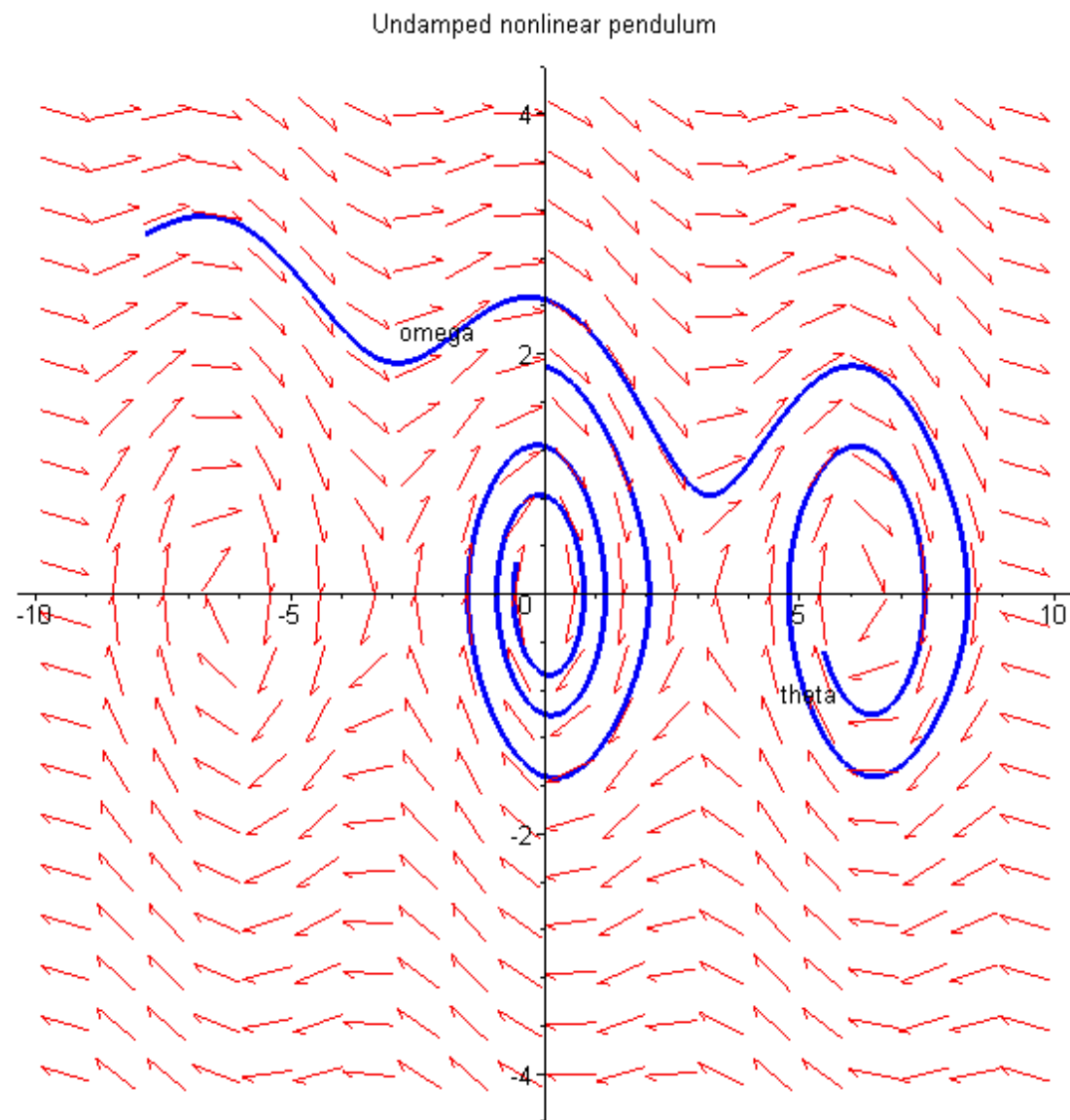
$$\beta = \frac{1}{8}$$

$$\frac{d}{dt} \theta(t) = \omega(t), \quad \frac{d}{dt} \omega(t) = -\sin(\theta(t)) - \frac{1}{8} \omega(t)$$

$$\theta(0) = -\frac{5\pi}{2}, \quad \omega(0) = 3$$

$$\theta(0) = 0, \quad \omega(0) = \frac{19}{10}$$

```
> a:=3*Pi;
b:=4;
DEplot([dde],[vars],t=0..20,[[dic1],[dic2]],theta=-a..a,omega=-b..b,
stepsize=.05,method=rkf45,
title=`Undamped nonlinear pendulum`,
color=red,linecolor=blue,arrows=small,thickness=3);
```



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### The energy as a function of the state

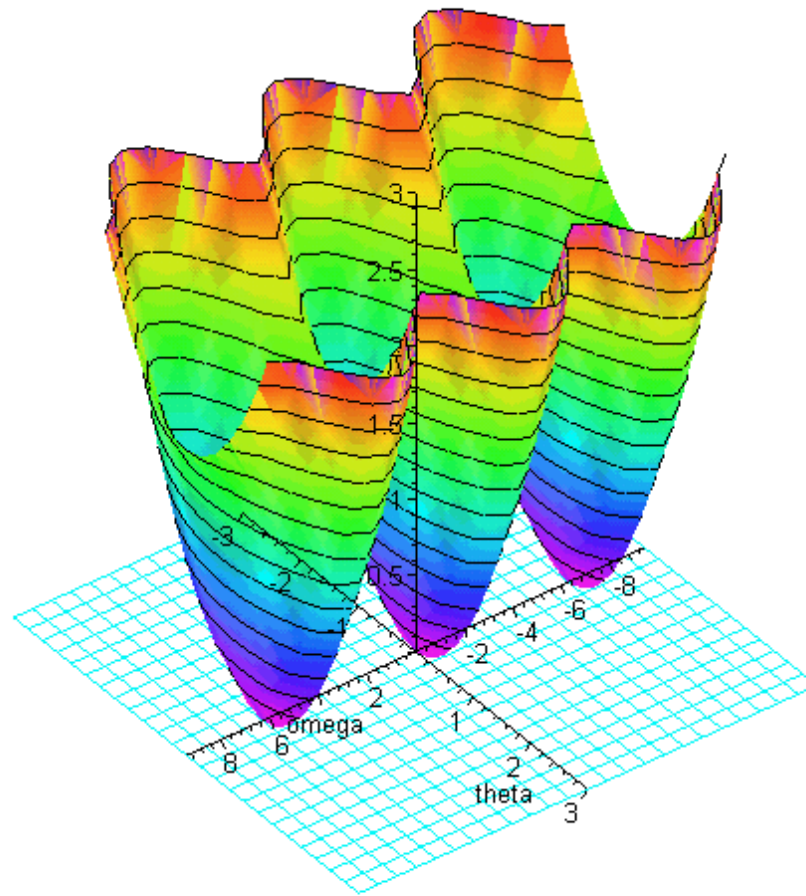
> **V:=omega^2/2+(1-cos(theta));**

$$V = \frac{\omega^2}{2} + 1 - \cos(\theta)$$

> **a:=3\*Pi;**  
**b:=3;**  
**display([**  
**plot3d(V,theta=-a..a,omega=-b..b,style=patchcontour,shading=zhue)**  
**,plot3d(0,theta=-a..a,omega=-b..b,style=wireframe,color=cyan)**  
**],axes=normal,orientation=[53,53],view=[-a..a,-b..b,0..3]);**

$$a := 3\pi$$

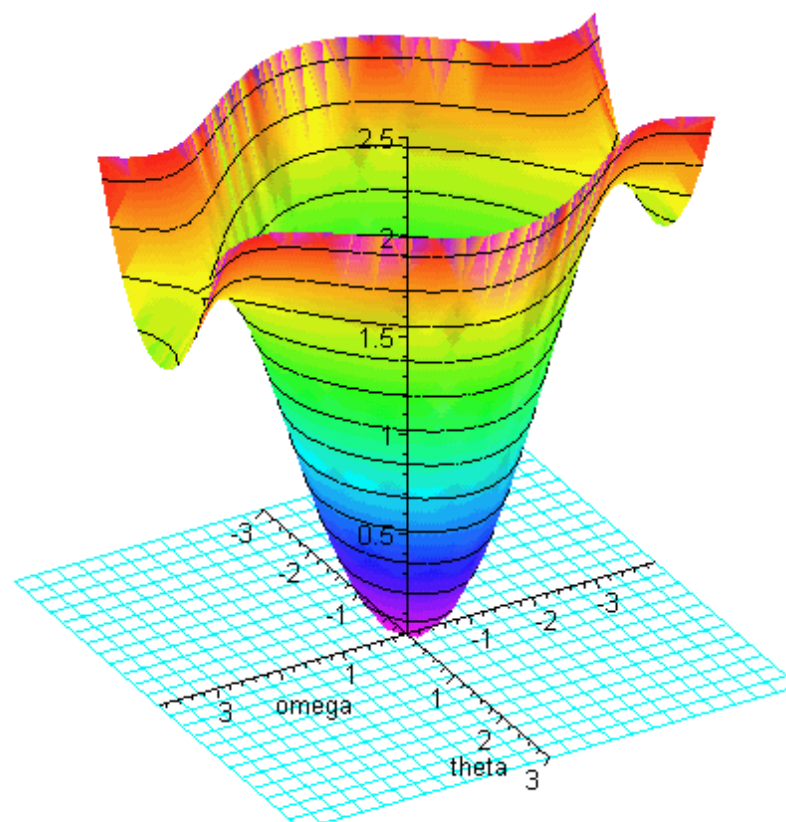
$$b := 3$$



```
> a:=5/4*Pi;
b:=3;
display([
  plot3d(V,theta=-a..a,omega=-b..b,style=patchcontour,shading=zhue)
  ,plot3d(0,theta=-a..a,omega=-b..b,style=wireframe,color=cyan)
],axes=normal,orientation=[60,60],view=[-a..a,-b..b,0..5/2]);
```

$$a := \frac{5\pi}{4}$$

$$b := 3$$



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&gt;

**Energy along trajectories as a function of time**

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> **Vt:=subs(theta=theta(t),omega=omega(t),V);**

$$Vt := \frac{1}{2} \omega(t)^2 + 1 - \cos(\theta(t))$$

> **usol1:=dsolve({ude,uic1},{vars},type=numeric);**  
**usol2:=dsolve({ude,uic2},{vars},type=numeric);**  
**dsol1:=dsolve({dde,dic1},{vars},type=numeric);**  
**dsol2:=dsolve({dde,dic2},{vars},type=numeric);**

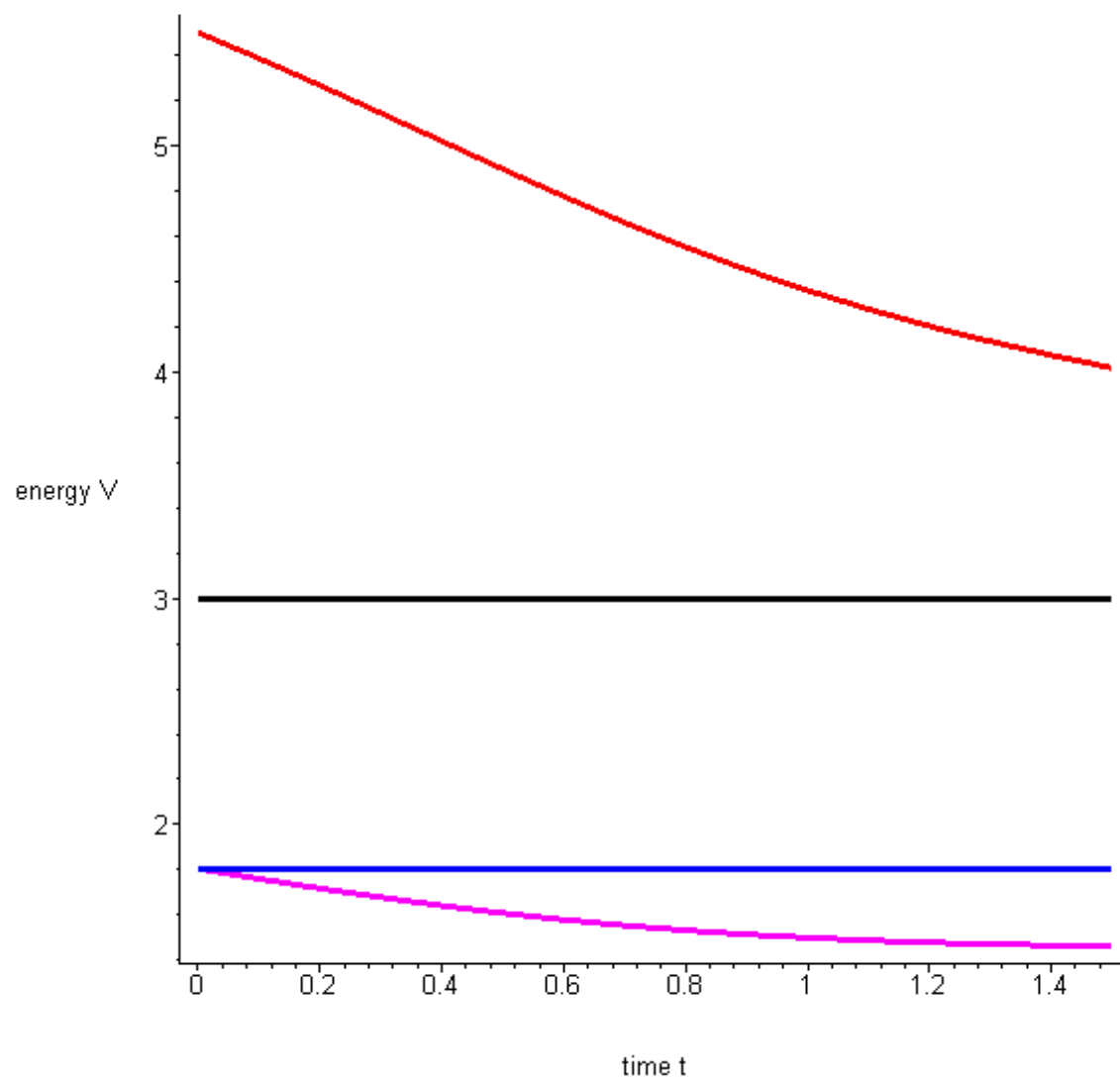
```
usol1 := proc(x_rkf45) ... end proc
```

```
usol2 := proc(x_rkf45) ... end proc
```

```
dsol1 := proc(x_rkf45) ... end proc
```

```
dsol2 := proc(x_rkf45) ... end proc
```

> **display(zip((sol,cc)->**  
**plot(map(tt->subs(sol(tt/10),[t,Vt]),[\$0..15]),color=cc,thickness=3),**  
**[usol1,usol2,dsol1,dsol2],[black,blue,red,magenta],**  
**title=`Energy along trajectories as a function of time`,**  
**titlefont=[TIMES,BOLD,16],**  
**axes=frame,**  
**labels=[`time t`,`energy V`]);**

**Energy along trajectories as a function of time**

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**Trajectories lifted to the energy surface**

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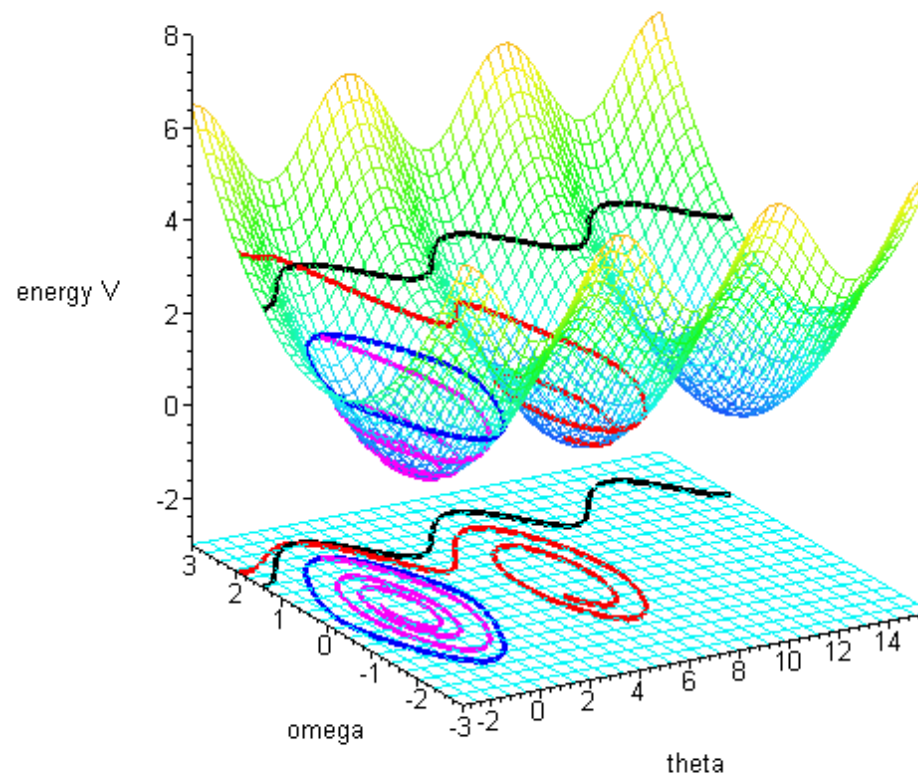
> **arange:=-Pi..5\*Pi;**

```

brange:=-3..3:
drop:=-3:
display(
[
op(
zip((sol,cc)->
op(
spacecurve(map(tt->subs(sol(tt/10),[theta(t),omega(t),Vt]),[$0..200]),color=cc,thickness=3),
spacecurve(map(tt->subs(sol(tt/10),[theta(t),omega(t),drop]),[$0..200]),color=cc,thickness=3)
]),
[usol1,usol2,dsol1,dsol2],[black,blue,red,magenta]
),
plot3d(drop,arange,brange,style=wireframe,color=cyan,grid=[30,20]),
plot3d(V,theta=arange,omega=brange,style=wireframe,shading=zhue,grid=[50,40]
],
orientation=[-120,70],
view=[arange,brange,drop..8],
title=`Integral curves and their lifts to the energy surface`,
titlefont=[TIMES,BOLD,16],
axes=frame,
labels=[`theta`,`omega`,`energy V`]);

```

### Integral curves and their lifts to the energy surface



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### Energy decay rate along trajectories

```

> DVt:=diff(Vt,t);
DVt:=subs(dde,diff(Vt,t));
DV:=subs(theta(t)=theta,omega(t)=omega,simplify(DVt));

```

$$DVt := \omega(t) \left( \frac{d}{dt} \omega(t) \right) + \sin(\theta(t)) \left( \frac{d}{dt} \theta(t) \right)$$

$$DVt := \omega(t) \left( -\sin(\theta(t)) - \frac{1}{8} \omega(t) \right) + \sin(\theta(t)) \omega(t)$$

$$DV := -\frac{\omega^2}{8}$$

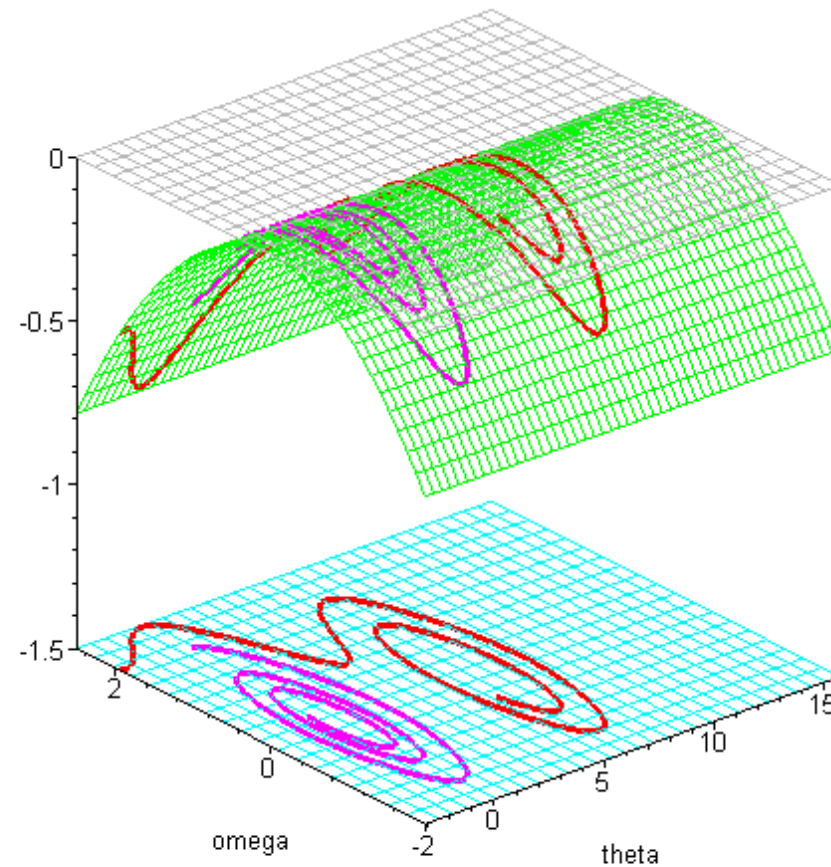


```

> arange:=-Pi..5*Pi:
brange:=-2..5/2:
drop:=-3/2:
display(
[
op(
zip((sol,cc)->
op([
spacecurve(map(tt->subs(sol(tt/10),[theta(t),omega(t),DVt]),[$0..200]),color=cc,thickness=3),
spacecurve(map(tt->subs(sol(tt/10),[theta(t),omega(t),drop]),[$0..200]),color=cc,thickness=3)
]),
[dsol1,dsol2],[red,magenta]
),
plot3d(0,arange,brange,style=wireframe,color=gray,grid=[30,20]),
plot3d(drop,arange,brange,style=wireframe,color=cyan,grid=[30,20]),
plot3d(DV,theta=arange,omega=brange,style=wireframe,color=green,grid=[50,40])
],
orientation=[-130,65],
view=[arange,brange,drop..0],
title=`The decay dV/dt is NOT strictly negative`,
titlefont=[TIMES,BOLD,16],
axes=frame,tickmarks=[3,3,3],
labels=[`theta`,`omega`,`decay DV/dt`]);

```

**The decay  $dV/dt$  is NOT strictly negative**



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