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Computer Controlled Systems

1st midterm test

2019. 10. 22.

computational problems (25 points)

(The answers can be given in Hungarian)

In case of yes-no questions, please, always justify your answer!

The test-sheet should be submitted with the computational part (white papers)!

1. We consider a SISO LTI system given by the following state-space model $(\Sigma = 10pt)$

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t), \end{cases} \tag{i}$$

where $A = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ -1 & 0 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}$, $D = 0$

$$A^2 = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 3 & 0 & 1 \end{pmatrix} \tag{ii}$$

(a) Give a possible state value, which can not be reached using any input functions.

$$\mathcal{C}_3 = \begin{pmatrix} 1 & -2 & 4 \\ 0 & 0 & 0 \\ 1 & -2 & 4 \end{pmatrix}, \ \mathrm{Im}\{\mathcal{C}_3\} = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$
$$\text{e.g. } x = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ cannot be reached.} \tag{iii}$$

(b) Compute the unobservable subspace of system (i). (2p)

$$\mathcal{O}_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 9 & 0 \end{pmatrix}, \ \mathrm{Ker}\{\mathcal{O}_3\} = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}. \tag{iv}$$

(c) Is the state-space model (A, B, C) minimal? (2p)
It is not jointly controllable and observable, thus it is not minimal.

(d) Is the system globally asymptotically stable? (2p)
Eigenvalues of A are: 1, -2, -3. It is not asymptotically stable.

(e) Note that the controllable subspace is contained in the unobservable subspace. Then, try to predict how the output is affected by the input.
Hint. You can also compute the transfer function $H(s)$ but it is not necessary.

$$H(s) = 0, \text{ the input does not have any affect on the output.} \tag{v}$$

2. It is given the following second order linear differential equation: $(\Sigma = 10pt)$

$$\ddot{y}(t) + 6\dot{y}(t) + 8y(t) = 3u(t), \tag{vi}$$

where $u(t)$ is the input signal, $y(t)$ is the output signal.

(a) Determine the transfer function $H(s)$ for this system. (2pt)

$$H(s) = \frac{3}{s^2 + 6s + 8} \tag{vii}$$

(b) Is the system BIBO stable?

Poles: -2, -4. The system is BIBO stable.

(c) Solve the differential equation (vi), if the input is $u(t) = e^{-t} + 5e^{t}$ ($y(t) =$?) and the initial conditions are $y(0) = \dot{y}(0) = 2$. (4pt)

After the Laplace transformation:

$$\begin{aligned}
&(s^2 + 6s + 8)Y(s) - 2s - 14 = \frac{3}{s+1} + \frac{15}{s-1} = \frac{18s + 12}{s^2 - 1} \\
&\Rightarrow Y(s) = \frac{2s^3 + 14s^2 + 16s - 2}{(s-1)(s+1)(s+2)(s+4))} = \frac{1}{s-1} + \frac{1}{s+1} - \frac{1}{s+2} - \frac{1}{s+4}
\end{aligned}$$

(d) Compute the DC-gain of system (vi). (2pt)

Hint. DC-gain $= \lim_{t\to\infty} y(t)$ if the input $u(t) = 1(t)$ is the unit step function. Furthermore, $\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s)$, where $Y(s)$ is the Laplace transform of $y(t)$. $\mathcal{L}\{1(t), s\} = \frac{1}{s}$.

DC gain: $H(0) = \frac{3}{8}$.

3. We consider the mass-spring-damper system $(\Sigma = 5pt)$

$$m\ddot{y} + D\dot{y} + ky^3 = 0, \Rightarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1^3 - x_2 \end{cases} \Rightarrow f(x) = \begin{pmatrix} x_2 \\ -x_1^3 - x_2 \end{pmatrix}. \tag{viii}$$

with a nonlinear spring characteristics $F_k(y) = -ky^3$. Let $m = 1$, $D = 1$, $k = 1$. We consider a Lyapunov function of the form:

$$V(x) = \frac{(x_1 + x_2)^2}{2} + ax_1^4, \text{ where } x_1 = y,\ x_2 = \dot{y}. \tag{ix}$$

Coefficient a is a free variable and is meant to be selected such that $V(x)$ is a Lyapunov function for (viii). Determine the value of a such that $V(x)$ satisfies the Lyapunov conditions (i.e. with $V(x)$ we can prove global stability for system (viii)).

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}V(x) &= \left\langle \frac{\partial V}{\partial x}, \dot{x} \right\rangle = \left\langle \frac{\partial V}{\partial x}, f(x) \right\rangle = \begin{pmatrix} x_1 + x_2 + 4ax_1^3 & x_1 + x_2 \end{pmatrix} \begin{pmatrix} x_2 \\ -x_1^3 - x_2 \end{pmatrix} \\
&= 4ax_1^3 x_2 - x_1^4 - x_1^3 x_2 = -x_1^4 \le 0 \text{ for all } x \in \mathbb{R}^2 \text{ if } a = 0.25.
\end{aligned} \tag{x}$$