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Computer Controlled Systems

1st midterm test

2019. 10. 22.

computational problems (25 points)

(The answers can be given in Hungarian)

In case of yes-no questions, please, always justify your answer!

The test-sheet should be submitted with the computational part (white papers)!

1. We consider a SISO LTI system given by the following state-space model ($\Sigma = 10pt$)

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t), \end{cases} \tag{i}$$

where $A = \begin{pmatrix} 1 & 0 & 0 \\ -3 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}$, $D = 0$

$$A^2 = \begin{pmatrix} 1 & 0 & 0 \\ -3 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix} \tag{ii}$$

(a) Compute the controllable subspace of system (i). (2p)

$$\mathcal{C}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & -2 & 4 \\ 1 & 0 & 0 \end{pmatrix}, \ \mathrm{Im}\{\mathcal{C}_3\} = \mathrm{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}. \tag{iii}$$

(b) Give two different state values, which can not be distinguished from each other by measuring only $y(t) = Cx(t)$.

$$\mathcal{O}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & -2 & 4 \\ 1 & 0 & 0 \end{pmatrix}, \ \mathrm{Ker}\{\mathcal{O}_3\} = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

$$\text{e.g. } x_1 = \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix} \in \mathrm{Ker}\{\mathcal{O}_3\}, \ x_2 = \begin{pmatrix} 3 \\ -3 \\ 1 \end{pmatrix} \in \mathrm{Ker}\{\mathcal{O}_3\}. \tag{iv}$$

(c) Is the state-space model (A, B, C) minimal? (2p)

It is not jointly controllable and observable, thus it is not minimal.

(d) Is the system globally asymptotically stable? (2p)

Eigenvalues of A are: 1, -2, -3. It is not asymptotically stable.

(e) Is the system BIBO stable? (2p)

$$\mathrm{adj}(sI - A) = \begin{pmatrix} (s+2)(s+3) & -3(s+3) & 0 \\ 0 & (s-1)(s+3) & 0 \\ 0 & 0 & (s-1)(s+2) \end{pmatrix}^{\top}.$$

$$(sI - A)^{-1} = \begin{pmatrix} \frac{1}{s-1} & 0 & 0 \\ \frac{-3}{(s-1)(s+2)} & \frac{1}{s+2} & 0 \\ 0 & 0 & \frac{1}{s+3} \end{pmatrix} \Rightarrow H(s) = C(sI - A)^{-1}B = \frac{1}{s+2}. \tag{v}$$

therefore $h(t) = e^{-2t}$, which is absolute integrable on $\mathbb{R}_+$. It is BIBO stable.

2. It is given the following state space model: $(\Sigma = 10pt)$

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -8x_1(t) - 6x_2(t) + 3u(t) \end{cases} \quad \text{with } y(t) = x_1(t) \tag{vi}$$

where $u(t)$ is the input signal, $y(t)$ is the output signal.

(a) Determine the model matrices (A, B, C) of system (vi). (1p)

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}, \text{ where } A = \begin{pmatrix} 0 & 1 \\ -8 & -6 \end{pmatrix}, \ B = \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \ C = \begin{pmatrix} 1 & 0 \end{pmatrix}. \tag{vii}$$

(b) Determine the transfer function $H(s)$ for this system. (2p)

$$H(s) = C(sI - A)^{-1}B = C\begin{pmatrix} s & -1 \\ 8 & s+6 \end{pmatrix}^{-1} B = \frac{1}{s^2 + 6s + 8}\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} s+6 & 1 \\ -8 & s \end{pmatrix}\begin{pmatrix} 0 \\ 3 \end{pmatrix} = \frac{3}{s^2 + 6s + 8},$$

(c) Using Laplace transformation determine the value of $Y(s) = \mathcal{L}\{y(t), s\}$ if the input is $u(t) = -2e^{-3t}$ and the initial state values are $x_1(0) = 0$ and $x_2(0) = 5$. (4p)

$$Y(s) = C(sI - A)^{-1}x(0) + C(sI - A)^{-1}BU(s) = C(sI - A)^{-1}\begin{pmatrix} 0 \\ 5 \end{pmatrix} + H(s)\frac{2}{s+3} = \frac{5s + 9}{(s+2)(s+3)(s+4)}$$

(d) Compute the output $y(t)$ if the input and the initial state values are the same as in the previous point. (2p)

$$Y(s) = \frac{6}{s+3} - \frac{1}{2(s+2)} - \frac{11}{2(s+4)} \Rightarrow y(t) = 6e^{-3t} - \frac{1}{2}e^{-2t} - \frac{11}{2}e^{-4t} \tag{viii}$$

3. It is given the following nonlinear state-space model $(\Sigma = 5pt)$

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = x_3^2 - x_1, \\ \dot{x}_3 = -2x_3(x_3^2 - x_1), \end{cases} \tag{ix}$$

and a Lyapunov function candidate:

$$V(x) = \frac{x_2^2}{2} + \frac{(x_3^2 - x_1)^2}{2} > 0 \text{ for all } x \neq 0, \text{ and } V(x) \text{ is cont. differentiable.} \tag{x}$$

Check, whether $V(x)$ satisfies the Lyapunov conditions (i.e. with $V(x)$ we can prove global stability for system (ix)).

$$\begin{aligned} \frac{\partial V}{\partial x} &= \begin{pmatrix} -(x_3^2 - x_1) & x_2 & 2(x_3^2 - x_1)x_3 \end{pmatrix}, \ f(x) = \begin{pmatrix} x_2 \\ x_3^2 - x_1 \\ -2x_3(x_3^2 - x_1) \end{pmatrix}. \\ \frac{\mathrm{d}}{\mathrm{dt}}V(x) &= \left\langle \frac{\partial V}{\partial x}, \dot{x} \right\rangle = \left\langle \frac{\partial V}{\partial x}, f(x) \right\rangle = -4\Big(x_3(x_3^2 - x_1)\Big)^2 \leq 0 \text{ for all } x \in \mathbb{R}^3. \end{aligned} \tag{xi}$$