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Application of the Chebyshev polynomial in solving Fredholm integral equations

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ABSTRACT

In this paper the Fredholm integral equation of the second kind is solved, where Chebyshev polynomials are applied to approximate a solution for an unknown function in the Fredholm integral equation and convert this equation to a system of linear equations. Also, convergence and rate of convergence are given. The accuracy of this method is verified through some numerical examples and the results are compared to a previous result set to investigate the effects of choosing Chebyshev and Legendre polynomials.

1. Introduction

Given the Fredholm integral equation of the second kind:

$$\varphi(s) = f(s) + \int_a^b K(s,t)\varphi(t)\mathrm{d}t, \quad -\infty < a \le s \le b < \infty \tag{1}$$

in solving the integral equation with a given kernel $K(s,t)$ and the function $f(s)$, the problem is typically to find the function $\varphi(t)$. Maleknejad et al. [1] estimated $\varphi(t)$ using Legendre polynomials and solved the integral equation. In this study, Chebyshev polynomials are employed to approximate the $\varphi(t)$, and the effects of the Chebyshev polynomials on the accuracy of the estimation of $\varphi(t)$ are compared through several numerical problems.

2. Discretization of integral equation

In this section, Eq. (1) is discretized and converted to a system of linear equations. Chebyshev polynomials are chosen as basis functions to estimate the solution of the integral equation, $\varphi(t)$, together with the collocation method.

The Chebyshev polynomials with the interval of orthogonality $[-1, 1]$ are defined as [2]:

$$C_{n+1}(x) = 2xC_n(x) - C_{n-1}(x), \quad \text{and} \quad C_0(x) = 1; \qquad C_1(x) = x. \tag{2}$$

Proposition 1. *Let $x(t) \in H^k(-1,1)$ (Sobolev space), $T_n(x(t)) = \sum_{i=0}^{n} a_i C_i(t)$ be the best approximation polynomial of $x(t)$ in L_2-norm, and the truncation error:*
$||x(t) - T_n(x(t))||_{L_2[-1,1]} \le C_0 m^{-k} ||x(t)||_{H^k(-1,1)}$, *where C_0 is a positive constant, which depends on the selected norm and is independent of $x(t)$ and m; m is the degree of Chebyshev polynomials (Proof [3]).*

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At first, we estimate the unknown function $\varphi(t)$ with the Chebyshev polynomials as

$$\varphi(t) \approx T_n(\varphi(t)) = \sum_{i=0}^{n} a_i C_i(t). \tag{3}$$

Substituting (3) into (1) and get

$$\sum_{i=0}^{n} a_i C_i(s) = f(s) + \int_a^b K(s,t) \sum_{i=0}^{n} a_i C_i(t) \mathrm{d}t. \tag{4}$$

Hence the residual equation is defined as

$$R_n(s) = \sum_{i=0}^{n} a_i C_i(s) - \int_a^b K(s,t) \sum_{i=0}^{n} a_i C_i(t) \mathrm{d}t - f(s). \tag{5}$$

The unknown coefficients a_i are defined by selecting several collocation points s_j so that $R_n(s_j) = 0$ for j from 0 to n. In this study the collocation points are evenly selected from the space $[a, b]$ that

$$s_j = a + \frac{j(b-a)}{n}, \quad j = 0, \ldots, n. \tag{6}$$

Thus, this integral equation (Eq. (4)) can be converted to a system of linear equations $A_n X = b_n$ where

$$\begin{aligned} A_n &= \left[C_i(s_j) - \int_a^b K(s_j,t) C_i(t) \mathrm{d}t \right]_{j=1}^{n}, \quad i = 0, \ldots, n \\ b_n &= [f(s_j)], \quad j = 0, \ldots, n \\ X^T &= [a_i]_{i=0}^{n}. \end{aligned} \tag{7}$$

Theorem 1. *Let χ be a Banach space, let κ be a bounded operator from χ into χ, and for a nonzero λ, if $\lambda - \kappa : \chi \to \chi$ is one to one and onto, then we have*

$$||\kappa - T_n\kappa|| \to 0 \quad \text{as } n \to \infty. \tag{8}$$

Furthermore, for all sufficiently large n, say $n \geq N$, the operator $(\lambda - T_n\kappa^{-1})$ exists as a bounded operator from χ to χ and is uniformly bounded

$$\sup_{n \geq N} ||(\lambda - T_n\kappa)^{-1}|| < \infty. \tag{9}$$

For the solution of $(\lambda - T_n\kappa)x_n = T_n y, x_n \in \chi$ and $(\lambda - \kappa)x = y$ we can have

$$\begin{aligned} &x - x_n = \lambda(\lambda - T_n\kappa)^{-1}(x - T_n(x)) \\ &\frac{|\lambda|}{||\lambda - T_n\kappa||} ||x - T_n(x)|| \leq ||x - x_n|| \leq |\lambda| ||(\lambda - T_n\kappa)^{-1}|| ||x - T_n(x)||. \end{aligned} \tag{10}$$

(Proof [4]).

From Proposition 1, it is concluded that approximation rate of Chebyshev polynomials is m^{-k}, and Theorem 1 indicates that $||x - x_n||$ converge to zero at exactly the same speed as $||x - T_n(x)||$.

3. Numerical examples

In this section, several numerical examples of the Fredholm integral equation (Eq. (1)) are considered to show the accuracy of presented method. In this study, all examples are solved using the method stated in Section 2, and the integral equations are converted to systems of linear equations following Eqs. (6) and (7). All calculations are performed using Maple 11 and MatLab; the detailed steps are:

1. Construct $n \times n$ square matrix A_m (from Eq. (7)).
2. Build up $n \times 1$ vector b_m (from Eq. (7)).
3. Calculate X using $X = A_m^{-1} b_m$ and determine all a_i.
4. Estimate $\varphi(t)$ based on the a_i using Eq. (3).
5. Compare the approximated $\varphi(t)$ to the exact one and show in figures.

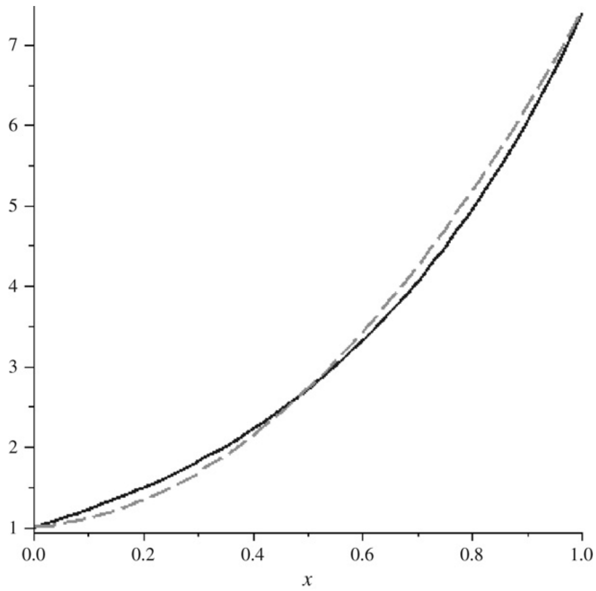

Fig. 1. Result of Example 1 for $n = 2$.

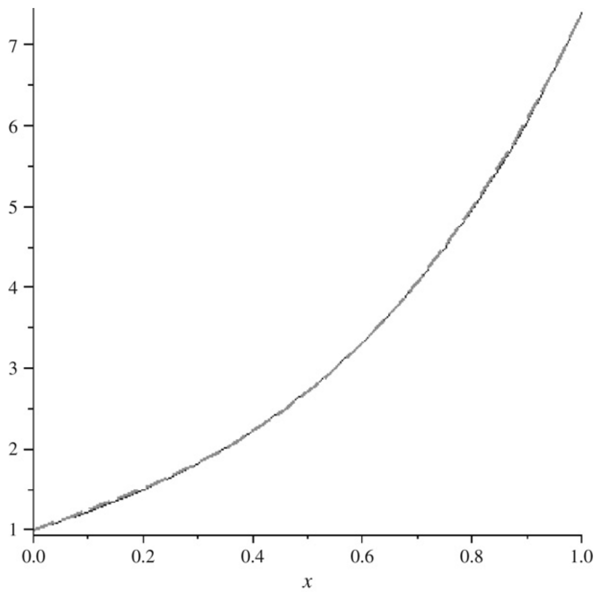

Fig. 2. Result of Example 1 for $n = 3$.

Numerical results are compared with the exact solutions and plotted in following figures to illustrate the efficiency of the proposed method. (In all figures the exact solutions are represented by solid lines while the numerical solutions are plotted with dashed lines.)

Example 1. Solve Eq. (1) with $a = -1, b = 1$ and

$$K(s,t) = e^{(2s - \frac{5}{3}t)}$$
$$f(s) = e^{(2s + \frac{1}{3})}$$

where the exact solution is $\varphi(t) = e^{2t}$ and results are shown in Figs. 1 and 2 with different n.

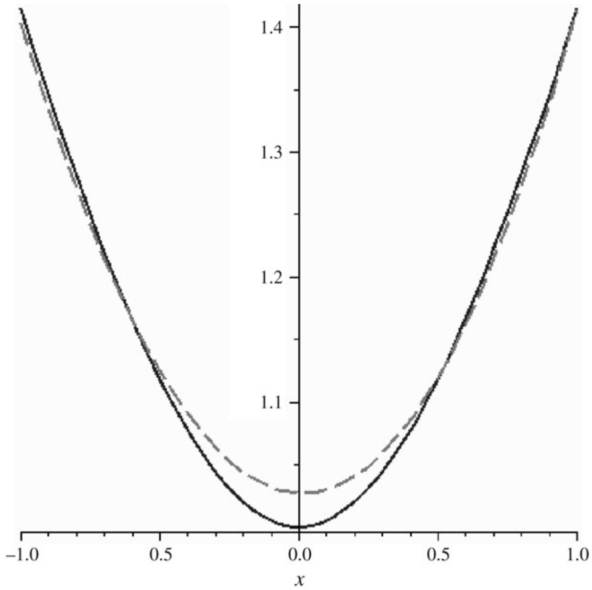

Fig. 3. Result of Example 2 for $n = 3$.

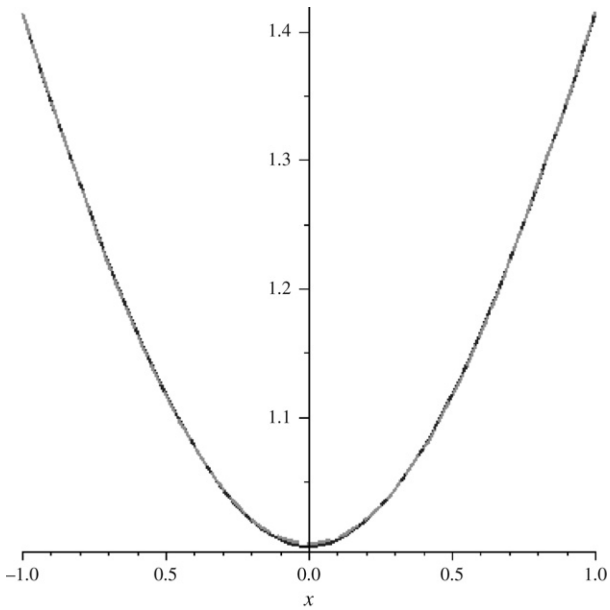

Fig. 4. Result of Example 2 for $n = 5$.

Example 2. Solve Eq. (1) with $a = -1, b = 1$ and

$$K(s,t) = \frac{(s-t)^3}{s^2(1+t^2)}$$

$$f(s) = \sqrt{1+s^2} - \frac{3(\sqrt{2} - \arcsin h(1))}{s} - 2s \arcsin h(1)$$

where the exact solution is $\varphi(t) = (1+t^2)^{1/2}$ and results are shown in Figs. 3 and 4 with different n.

Comparing the Figs. 3 and 4 to the results obtained by Maleknejad et al. [5], it is proved that both Chebyshev and Legendre polynomials can be used to successfully solve the Fredholm integral equation of the second kind. Comparatively, Legendre polynomials are easier to be applied in such problems because such polynomials have a unit weight function. However, as concluded by Mason and Hanscomb [6], the partial sums of a first kind Chebyshev expansion of a continuous function in $[-1, 1]$ converge faster than the partial sums of an expansion in any other orthogonal polynomials. Therefore, the Chebyshev polynomials usually yield better estimation of the unknown function $\varphi(t)$ when being used for solving such problems.

4. Conclusion

In this paper, the Fredholm integral equation of the second kind is solved by employing Chebyshev polynomials and the collocation method. Convergence of the presented method and its convergence rate are proved in Proposition 1 and Theorem 2. As shown in the figures, the proposed method provides good efficiency so that in order to acquire enough accuracy, we only need to convert the integral equation to the system of linear equations by an order less than five. Besides the Legendre method presented by Maleknejad et al. [1], this paper proves that the Chebyshev polynomials can also be used to solve the Fredholm integral equation of the second kind with high accuracy and efficiency. Apparently, in order to have the convergence of the present method, both kernel functions $K(s, t)$ and $f(s)$ have to be continuous.

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