Functional analysis

Exercise Problems

14.03.2018.

Metric space, Normed space, Inner product space

N1. Check whether the following formulas define metric on $\mathbb{R}$:

a) $d(x,y) = \big|\ |x| - |y|\ \big|$

b) $d(x,y) = \sqrt{|x-y|}$

c) $d(x,y) = |\operatorname{arc\,tg} x - \operatorname{arc\,tg} y|$

N2. Check whether the following formulas define metric in $\mathbb{R}^2$ ($x = (x_1, x_2)$ és $y = (y_1, y_2)$):

a) $d(x,y) = \max(|x_1 - y_1|, |x_2 - y_2|)$,

b) $d(x,y) = \min(|x_1 - y_1|, |x_2 - y_2|)$,

N3. Verify that the following $\|\cdot\|$ defines a norm in $\mathbb{R}^2$ ($a, b \epsilon \mathbb{R}$ are fixed parameters):

$$\|x\| = \sqrt{\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2}}, \quad x = (x_1, x_2) \epsilon \mathbb{R}^2.$$

Sketch the open unit ball in $\mathbb{R}^2$ with respect to this norm, when

a) $a = 2$, $b = 1$,

b) $a = 3$, $b = \frac{1}{3}$.

N4. (Test exercise in 2015.) Let V be the space of 3×2 matrices. Verify that the following formula defines a norm in this linear space:

$$\|A\| := \max_{i=1,2,3}(|a_{i1}| + |a_{i2}|), \qquad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} \epsilon \mathbb{R}^{3\times 2}.$$

N5. (Test exercise in 2015.) Let V be the space of 2×4 matrices. Verify that the following formula defines a norm in this linear space:

$$\|A\| := \max_{i=1,2,3,4} (|a_{1i}| + |a_{2i}|), \qquad A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix} \epsilon \mathbb{R}^{2\times 4}.$$

N6. (Exam exercice from 2016) Do these funtions define norms on $\mathbb{R}^2$? Verify.

$$\|(x,y)\|_a = \Big| \, |x| + 5\,|y| \, \Big|, \qquad \|(x,y)\|_b = \Big| \, |x| - 5\,|y| \, \Big|, \qquad (x,y)\epsilon\mathbb{R}^2.$$

If it is a norm, some more questions:

a) What is the induced metric?

b) Sketch the open unit ball in $\mathbb{R}^2$ with respect to this norm.

N7. Consider the linear space of continuously differentiable functions defined on [0,1] , denoted by $C^1[0,1]$ terét. Which of the following formulas give norme on this vector space?

a) $N_1(f) = \max_{x\epsilon[0,1]} |f(x)|,$

b) $N_2(f) = \max_{x\epsilon[0,1]} |f'(x)|,$

c) $N_3(f) = \max_{x\epsilon[0,1]} |f(x)| + \max_{x\epsilon[0,1]} |f'(x)|,$

d) $N_4(f) = |f(0)| + \max_{x\epsilon[0,1]} |f'(x)|\,.$

N8. (Test exercise in 2014.) In ℓ^1 let us consider sequences that have only finite number of nonzero element. Verify that they form a dense subset in ℓ^1-ben.

(+ Is a similar statement true in ℓ^∞?)

N9. (Test exercise in 2012.) Consider the following sequence space:

$$H = \{(x_n) \mid \sum_{n=1}^{\infty} x_n^2 (1+n^2) < \infty\}.$$

a) Show that it is az inner product on H: $\langle x,y\rangle_H := \sum_{n=1}^{\infty} x_n y_n (1+n^2)$.

b) Verify that $H \subset \ell^2$.

c) Verify for the induced norm: $\|x\|_H \geq \|x\|_2$, ha $x \epsilon H$.

N10. Recall the definition of the sequence spaces c, c_0, ℓ_1, ℓ_2, ℓ_∞ and consider the sequences $x = (x_n)$, $n = 1,2,\ldots$ (denoted also by $\{x_n\}_{n=1}^{\infty}$ or $\mathbf{x} =$ $= (x_1, x_2, \ldots)$ or $\mathbf{x} = (x_1, x_2, \ldots, x_n, \ldots)$ etc.)

$$(a)\ x_n = (-1)^n\,,\quad (b)\ x_n = \frac{1}{\sqrt{n}n}\,,\quad (c)\ x_n = (-1)^n \frac{1}{2^n}\,,\quad (d)\ x_n = \frac{\sqrt[n]{2}}{n^2}\,,\qquad n = 1,2,\ldots.$$

Do they belong to c, c_0, ℓ_1, ℓ_2, ℓ_∞ or not?

N11. Compute the norms of the sequences

$$(a)\ x_n = 1\,,\quad (b)\ x_n = \frac{1}{n}\,,\quad (c)\ x_n = (-1)^n \frac{1}{n}\,,\quad (d)\ x_n = \frac{1}{n^2}\,,\qquad n = 1,2,\ldots$$

in the sequence spaces c, c_0, ℓ_1, ℓ_2, and ℓ_∞.

N12. Point out that the standard $||\cdot||_\infty$ norm on dbR^n cannot be induced by a scalar product.

N13. Point out that the standard $||\cdot||_\infty$ norm on the sequence space ℓ^∞ cannot be induced by a scalar product.

N14. Set $f(x) = x$ and $g(x) = x^2$ whenever $x\epsilon[0,1]$. Compute the distance between functions f and g with respect to the

(a) maximum–norm on $C[0,1]$, (b) the L_1–norm on $C[0,1]$, (c) the L_2–norm on $C[0,1]$

Toplogy in metric spaces

T1. (N, d_N) is a metric space, $A \subset N$ is an open set. $a \epsilon A$ is an arbitrary point. Show that $A \setminus \{a\}$ is an open set.

T2. Let $f_n, f \epsilon C[a,b]$. Show that if $f_n \to f$ in $\|\cdot\|_{\max}$ norm, then $f_n \to f$ in $\|\cdot\|_1$ norm too. Is the converse statement true?

T3. (N, d_N) is a metric space, $K \subset N$ has finite number of elements. Verify that K is compact.

T4. (+) Let (X,d) be a compact metric space, and let $\{U_i : i \epsilon I\}$ be an open cover of X. Verify that $\exists r > 0$, such that $\forall x \epsilon X$ we have $B_r(x) \subset U_i$ for some i.

T5. (+) $X = (0,1) \cup \{2\}$ with the usual metric $d(x,y) = |x - y|$. Which of the following sets are open or closed in this metric space:
$(0,1)$, $(0,1) \cup \{2\}$, $\{2\}$, $\left(0, \frac{1}{2}\right]$?

T6. (Test exercise in 2016.) Let $E \subset \mathbb{R}$, $E \neq \emptyset$, such that $m(E) = 0$.

a) Verify that $\text{int}\,(E) = \emptyset)$.

b) Is it possible, that E is open? If yes, give an example.

c) Is it possible, that E is closed? If yes, give an example.

T7. Let $\mathcal{S} = \{\mathbf{x} = (x_1, x_2, \ldots, x_n, \ldots) \epsilon \ell_2 \mid x_n = 0 \text{ for each } n > N = N(\mathbf{x})\}$. Is $\mathcal{S}$ a linear subspace of ℓ_2? Is it a closed subspace? Is it a complete subspace? Is it a dense subspace? Is $\mathcal{S}$ a Hilbert space? Can You construct a sequence in $\mathcal{S}$ without convergent subsequences? (A natural basis in $\mathcal{S}$ will do.)

T8. Prove that countable subsets of $\mathbb{R}$ are of measure zero.

T9. Recall that $C[0,1] = (C[0,1], ||\cdot||_{max})$ is complete. Consider now (a) $(C[0,1], ||\cdot||_2)$ and (b) $(C[0,1], ||\cdot||_1)$, the spaces of continuous real functions defined on the interval $[0,1]$ and equipped with the norms

$$(a) \quad ||f||_1 = \int_0^1 |f(x)|\, dx \quad \text{and} \quad (b) \quad ||f||_2 = \sqrt{\int_0^1 |f(x)|^2\, dx}\,, \quad f \epsilon C[0,1],$$

respectively. Prove that none of them is complete.

T10. Consider the sequence space ℓ^∞ and prove that its unit sphere $S_1 =$ $= \{x\epsilon\ell^\infty \ : \ ||x||_\infty \leq 1\}$ is closed but not compact. Is there a countable dense set in ℓ^∞?

Lebesgue-measure. Measurable functions.

M1. Let us consider the following set with the usual metric $d(x, y) = |x-y|)$:

$$H = \{x = \frac{j}{2^k} \ : \ k\epsilon\mathbb{N}, \quad 1 \leq j \leq 2^k\}.$$

a) Is it open? Is it closed? Is it compact?Ez a halmaz nyílt? Zárt? Kompakt?

b) Is H Lebesgue-measurable? If yes, compute it's measure.

M2. Let $E \subset \mathbb{R}$ be a null-set. Show that it has no interior point.

M3. Define $H = \{x = \frac{p}{q}\sqrt{2} \quad : \quad p < q\epsilon\mathbb{N}\}$. Is it measurable? If yes, compute it's measue.

M4. Let $E \subset \mathbb{R}$ be a set of measure zero. Is it possible that E contains an interval?

M5. $C \subset [0,1]$ is the Cantor set. Let us define:

$$C^{(2k)} := \{2k \cdot x \ : \ x\epsilon C\}, \qquad k = 1,2,\ldots.$$

and finally

$$C^{(\infty)} := \bigcup_{k=1}^{\infty} C^{(2k)}.$$

Is $C^{(\infty)}$ measurable? If yes, compute it's measure.

M6. Prove that the measurability of function $|f|$ is a consequence of the measurability of function f. Is also the reverse implication true?

M7. Check the following properties of characteristic functions:

$$(a)\ \ \chi_A \cdot \chi_B = \chi_{A\cap B}\ ,\quad (b)\ \ \chi_A + \chi_B - \chi_{A\cap B} = \chi_{A\cup B}\ ,\quad (c)\ \ |\chi_A - \chi_B| = \chi_{A\triangle B}\ ,\quad \forall A, B \subset \mathbb{R}^n.$$

Is it true or not:

1. A metric space is automatically a normed space.
2. ℓ_1 is a proper subspace of ℓ_2.
3. Dimension of ℓ_2 is 2.
4. Equipped with the Euclidean metric, $X = \mathbb{R}^2$ is separable.
5. Equipped with the discrete metric d_{discr}, $X = \mathbb{R}^2$ is not separable.