KONZ12

Homeworks

1. $\|x\| = \max\{|2x_1 - x_2|, |2x_1|\}$ $\quad x \in R^2 \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

① $\|x\| \geq 0$ ✓

② $\|x\| = 0 \iff x = 0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad |2x_1| = 0 \to x_1 = 0$ ✓

$|2x_1 - x_2| = 0 \Rightarrow x_2 = 0$

③ $\|\lambda \cdot x\| = |\lambda| \cdot \|x\|$ $\quad \lambda x = \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \end{pmatrix}$

$\|\lambda x\| = \max\{|2 \cdot \lambda x_1 - \lambda x_2|, |2 \cdot \lambda x_1|\} = \max\{|\lambda||2x_1 - x_2| + |\lambda||x_1 2|\} =$

$= |\lambda| \max\{\dots\} = |\lambda| \cdot \|x\|$ ✓

④ $\|x + y\| \leq \|x\| + \|y\|$ $\quad x + y = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix}$

$\max\{|2x_1 + 2y_1 - x_1 - y_2|, |2x_1 + 2y_1|\} =$

$= \max\{|(2x_1 - x_1) + (2y_1 - y_2)|, |2x_1 + 2y_1|\} \leq \max\{|2x_1 - x_2| + |2y_1 - y_2|, |2x_1| + |2y_1|\}$

$$\boxed{\max\{a+b, c+d\} \leq \max\{a, c\} + \max\{b, d\}}$$

$\leq \max\{\underbrace{|2x_1 - x_2|, |2x_1|}_{\|x\|}\} + \|y\|\}$

Unit Ball:

$$B_1(0) = \{ x \in \mathbb{R}^2 \mid \|x\| < 1 \}$$

$$\|x\| = \max \{ |2x_1 - x_2|, |2x_1| \} < 1$$

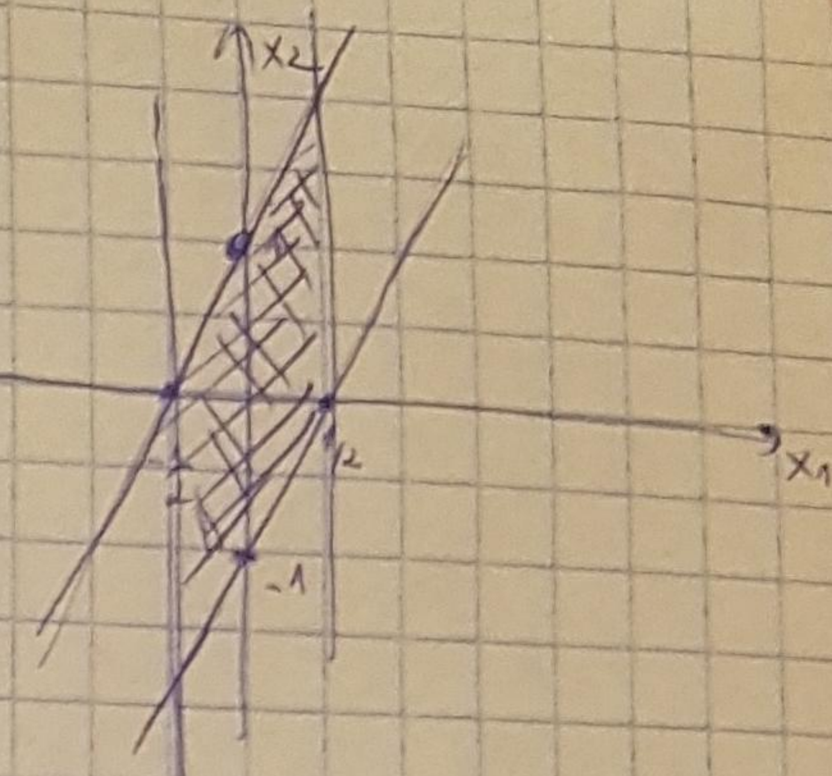
$$\rightarrow |2x_1| < 1 \qquad -\frac{1}{2} < x_1 < \frac{1}{2}$$

$$|2x_1 - x_2| < 1$$

$$\rightarrow -1 < 2x_1 - x_2 < 1$$

$$-1-2x < -x_2 < 1-2x \quad / (-1)$$

$$\underbrace{1+2x_1}_{f(x_1)} > x_2 > \underbrace{-1+2x_1}_{g(x_1)}$$



$f(x_1) = 1+2x$

$f(0) = 1$; $f(\frac{1}{2}) = 0$

$g(x_1) = -1+2x_1$

$g(0) = -1$

$g(\frac{1}{2}) = 0$

2. $\left.\begin{array}{l} f(x) = x \\ g(x) = x^2 \end{array}\right\}$ distance ? a) $\|f-g\|_2$ $\in C^1[0,1]$, $\|\bullet\|_p$ $\cancel{\|f\|_2 = \int_0^1}$

a) $\|f-g\|_2$

distance between f and g in $\|\bullet\|_2$

$$\rightarrow \|f-g\|_2 = \left(\underbrace{\int_0^1 [f(x) - g(x)]^2 dx}_{} \right)^{1/2} = \frac{1}{\sqrt{30}}$$

$$\int_0^1 (x - x^2)^2 dx = \cancel{\ldots} \int_0^1 (x^2 - 2x^3 - x^4) dx = \left[\frac{x^3}{3} - \frac{2x^4}{4} - \frac{x^5}{5} \right]_0^1 = \frac{1}{30}$$

$$\|f\|_\infty = \max_{x \in [0,1]} \{ f(x) \}$$

b) $\|f-g\|_\infty = \| \underbrace{x - x^2}_{h(x)} \| = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

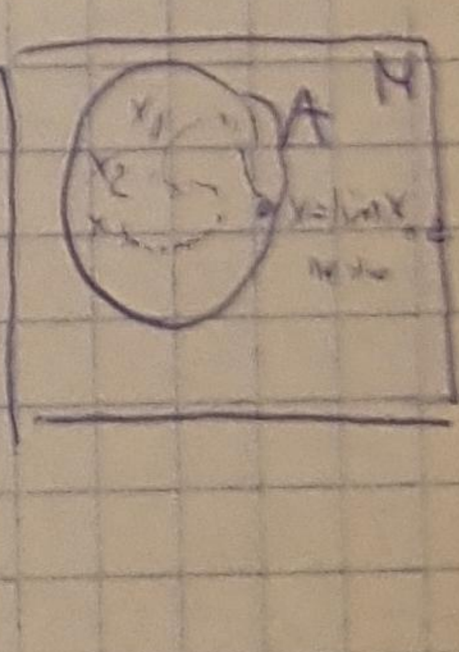
$\rightarrow$ maximum : $h'(x) = 1 - 2x = 0$

$x = \frac{1}{2}$

3. $\bar{B}(0) = \{ x \in \ell^1 \mid \|x\|_1 \leq 1 \}$

$\ell^1 = \{ x = (x_1, x_2 \ldots) \mid \sum_{k=1}^{\infty} |x_k| < \infty \}$ Prove That not Compact.

! Compact: (M, d) $A \subset M$ A is compact if $\forall (x_n) \subset A$ $\exists (x_{n_k})$ subsequence which has $\exists \lim_{n_k \to \infty} x_{n_k} \in A$.



// Részsorozatnak létezik A ban határértéke.

let $X_1 = (1, 0, 0, 0 \ldots)$ sequence of sequences.

$X_2 = (0, 1, 0, \ldots)$

$\vdots$

$X_n = (0, 0, \ldots \underbrace{1}_{n}, 0, \ldots)$

$X_n - X_m = (0, \ldots \overset{n}{1} \ldots \overset{m}{-1}, 0, 0)$

$\| X_n - X_m \| = 2$ → This can not be a Cauchy sequence.

Cauchy seq: $\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ such that $\| X_n - X_m \| < \varepsilon \ \forall n, m \geq N$

now ε can not be choosen any small....

If it is not a Cauchy sequence, it is not convergent!

$(x_n) \subset M$
(x_n) is conv $\Longrightarrow (x_n)$ Cauchy! if its not Cauchy → not convergent.

If the space is complete & (x_n) Cauchy $\Rightarrow (x_n)$ is conv.

$(x_n) \subset M$, (x_n) is conv

$\Downarrow$ $\Uparrow$ if M is complete.

(x_n) Cauch

$(C^1[0,1], \|\cdot\|_\infty)$ Complete Banach Space

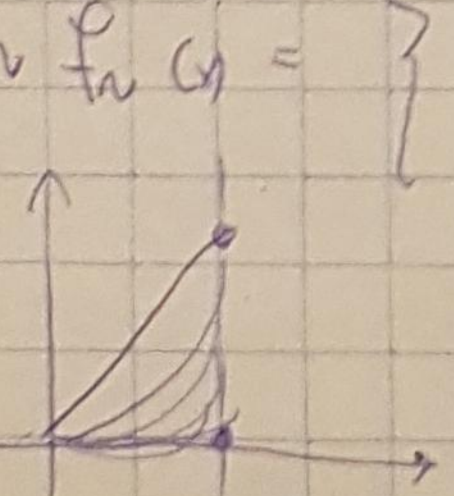
$C^1[0,1], \|\cdot\|_1)$ not complete → $f_n(x) = x^n$ this is a Cauchy seq.

Proof $n < m$,

$$\| f_n - f_m \| = \int_0^1 | f_n(x) - f_m(x) | \, dx = \int_0^1 | x^n - x^m | \, dx = \int_0^1 x^n - x^m \, dx =$$

$= \frac{1}{n} - \frac{1}{m}$ if $n, m \to \infty$ this $\frac{1}{n} - \frac{1}{m} \to 0$ Cauchy sequence.

$$\lim_{n \to \infty} f_n(x) = \begin{cases} 0, & x \in [0, 1) \\ 1, & x = 1 \end{cases}$$ not continous $\Rightarrow f \notin C^1[0,1]$

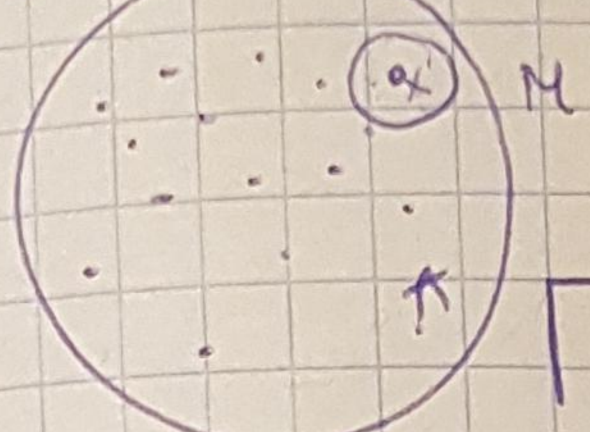


$\ell^\infty \to$ space of bounded sequences $\left\{ x = (x_1, x_2 \dots) \,\middle|\, \max_{i \in \mathbb{N}} |x_i| < \infty \right\}$

ℓ^∞ is not separable.

Def.: (M, d) $A \subset M$ A is dense in M if $\forall x \in M$ is a limit point of A.

formally $\forall \varepsilon > 0$ $B_\varepsilon(x) \cap A \neq \emptyset$



$A = \{a_1, a_2, a_3 \dots$

Separable def.

$$\boxed{\underline{M} \text{ is separable if } \exists A = \{a_1, a_2, a_3 \dots\} \overset{\subset}{\text{dense}} \text{ in } M}$$

let $A = \{a_1, a_2 \dots a_n \dots\}$ set of sequences

$\|x - y\|_\infty = \max_{i \in \mathbb{N}} |x_i - y_i|$

$a_1 = (a_{11}, a_{12}, a_{13} \dots)$

$a_2 = (a_{21}, a_{22}, a_{23} \dots)$

$x = (a_{11} + 1, a_{22} + 1, a_{33} + 1, \dots a_{nn} + 1, \dots)$

$d(x, a_i) = \|x - a_n\|_\infty \geq 1$ $\Rightarrow A$ cannot be dense in ℓ^∞ $\Rightarrow \ell^\infty$ is not separable

$\boxed{N10.}$ $c, c_0, \ell_1, \ell_2, \ell_\infty$ space of sequences

c: convergent sequences
c_0: 0 sequences
ℓ_1: space of sequences $\|x\|_1 < \infty$
ℓ_2: $\|x\|_2 < \infty$
ℓ_∞ bounded sequences. (Korlátos)

$x = (x_1, x_2, \dots x_n, \dots)$

a) $x_n = (-1)^n \in \ell_\infty$ in which space belong to?

b) $x_n = \frac{1}{n\sqrt{n}}$ $\to \lim_{n\to\infty} x_n \Rightarrow 0 \to$

c) $x_n = (-1)^n \frac{1}{2^n}$

d) $x_n = \frac{\sqrt[n]{2}}{n^2}$

a) $x_n = (-1)^n \in \ell^\infty$

b) $x_n = \frac{1}{n\sqrt{n}}$ $\to \lim_{n\to\infty} x_n = 0$, $\|x\|_1 = \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} < \infty$ convergent.

$\|x\|_2 = \sum_{n=1}^{\infty} \frac{1}{n^3} < \infty$

$\left.\begin{array}{l} \in \ell^1 \\ \in c, \in c_0 \\ \in \ell^2 \\ \in \ell^\infty \end{array}\right\} \underline{\underline{\text{All}}}$

c) $\in \ell^\infty, c, c_0, \ell_1, \ell_2$ All spaces. $(-1)^n \cdot \frac{1}{2^n}$ // $\sum \frac{1}{2^n} = 1$

d) $x_n = \frac{\sqrt[n]{2}}{n^2}$ $\to \boxed{\lim_{n\to\infty} \sqrt[n]{a} = 1}$

$\underbrace{}_{\frac{1}{\infty} \to 0}$ $\overset{?}{\in} c, c_0, \ell_1, \ell_2$ Proof: $\sum_{n=1}^{\infty} \frac{\sqrt[n]{2}}{n^2} < \sum_{n=1}^{\infty} \frac{2}{n^2} < \infty$

$\|x_1\| = \sum_{n=1}^{\infty} \frac{\sqrt[n]{2}}{n^2} \overset{?}{<} \infty$

Relations between: $C, C_0, \ell_1, \ell_2, \ell_\infty$

$\boxed{\ell_1 \subset \ell_2 \subset \ell_\infty}$!

→ ha ℓ_1-et bizonyítottam, akkor elme ℓ_2-nek és ℓ_∞-nek.