Functional analysis 2018 - Practical lecture

Practical lecture 10.

8th and 10th of May

Spectrum and eigenvalues of bounded linear operators.

1. Show that a linear operator $T \in B(X)$ is invertible if and only if, $Tx = 0$ implies $x = 0$.
2. (last week) Show that the linear operator $T\colon \ell^2 \to \ell^2$,
$$Tx = T(x_1, x_2, x_3, \dots) := \left(x_1, \frac{x_2}{2}, \frac{x_3}{3}, \dots, \frac{x_n}{n}, \dots\right)$$
is not invertible.
3. Determine the spectrum of the **left**-shift operator in ℓ^2. Which values of the spectrum are also eigenvalues of the operator?
4. Determine the spectrum of the **right**-shift operator in ℓ^2. Which values of the spectrum are also eigenvalues of the operator?
5. (last week) Determine the spectrum of the following linear operator $T\colon \ell^2 \to \ell^2$,
$$Tx = T(x_1, x_2, x_3, \dots) := \left(x_1, \frac{x_2}{2}, \frac{x_3}{3}, \dots, \frac{x_n}{n}, \dots\right)$$
What are the eigenvalues of T?

[HF$_1$] Determine the spectrum of the following linear operator $T\colon \ell^2 \to \ell^2$,
$$Tx = T(x_1, x_2, x_3, \dots) := \left(x_1, \frac{1}{2}\,x_2, \frac{2}{3}\,x_3, \dots, \frac{n-1}{n}\,x_n, \dots\right)$$
What are the eigenvalues of T?

6. Determine the norm and spectrum of the following linear operator $A\colon H \to H$, $Ax = \alpha x$, where H is a given Hilbert space and $\alpha \in \mathbb{R}$ is a constant.
7. Let $\mu\colon [0,1] \to \mathbb{R}$ be a fixed function in $C[0,1]$. Consider the following linear operator
$$A\colon C[0,1] \to C[0,1], \qquad (Ax)(t) := \mu(t)x(t), \qquad t \in [0,1].$$
Determine the spectrum of operator A if
   (a) $\mu(t) = t$;
   (b) $\mu \in C[0,1]$ is arbitrary

   Determine the eigenvalues of A in both cases.
8. Determine the spectrum of operator $T \in B(C[0,1])$, $(Tf)(x) = x^2 f(x)$.

(∗5) Show that the nonzero elements of $\sigma(AB)$ and $\sigma(BA)$ are the same, where $A, B \in B(H)$ and H is a given Hilbert space.

Bounded linear functionals. Dual space.

1. Show that for every linear functional $f : \mathbb{R}^n \to \mathbb{R}$ there exists a vector $a \in \mathbb{R}^n$, such that $f(x) = a^T x$.
2. Show that $c_0^* = \ell^1$.
3. Show that $(\mathbb{R}^n, \|\cdot\|_1)^* = (\mathbb{R}^n, \|\cdot\|_\infty)$.
4. Consider the sequence $(e^n)_{n\in\mathbb{N}}$ in ℓ^2 Hilbert space. Show that (e^n) is weakly convergent, but it is not convergent in the strong sense. $e_n = (0, 0, ..., 0, 1, 0, ...)$, i.e. only the nth element of e_n is nonzero.
5. Prove that in a given Hilbert space with an arbitrarily fixed $y \in H$, the linear functional $f_y(x) = \langle x, y\rangle$ is bounded and $\|f_y\| = \|y\|$.